LN18: Multivariate Normal Distribution

Yasutomo Murasawa

June 13, 2003

Contents

1 Density Function

Definition 1 *The density function of an* n-variate normal distribution *is* $\forall x \in \Re^n$,

$$f(x) := (2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right).$$

Remark: We write $\mathrm{N}(\mu, \Sigma)$.

Definition 2 *The* n-variate standard normal distribution *is* $\mathrm{N}(0, I_n)$.

2 Moment Generating Function

Theorem 1 *The mgf of* $\mathrm{N}(\mu, \Sigma)$ *is* $\forall t \in \Re^n$,

$$M(t) = \exp\left(\mu' t + \frac{t'\Sigma t}{2}\right).$$

Proof. We have

$$\begin{aligned} M(t) &:= \mathrm{E}\left(e^{t'X}\right) \\ &= \int_{-\infty}^{\infty} e^{t'x}(2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right) \mathrm{d}x \\ &= \int_{-\infty}^{\infty} (2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\left(t'x - \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right) \mathrm{d}x, \end{aligned}$$

where

$$
\begin{aligned}
& t'x - \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu) \\
= \; & t'x - \frac{1}{2}\left(x'\Sigma^{-1}x - 2\mu'\Sigma^{-1}x + \mu'\Sigma^{-1}\mu\right) \\
= \; & -\frac{1}{2}\left[x'\Sigma^{-1}x - 2\left(\mu'\Sigma^{-1/2} + t'\Sigma^{1/2}\right)\Sigma^{-1/2}x + \mu'\Sigma^{-1}\mu\right] \\
= \; & -\frac{1}{2}\left[\Sigma^{-1/2}x - \left(\Sigma^{-1/2}\mu + \Sigma^{1/2}t\right)\right]'\left[\Sigma^{-1/2}x - \left(\Sigma^{-1/2}\mu + \Sigma^{1/2}t\right)\right] \\
& +\mu't + \frac{t'\Sigma t}{2} \\
= \; & -\frac{1}{2}[x-(\mu+\Sigma t)]'\Sigma^{-1}[x-(\mu+\Sigma t)] + \mu't + \frac{t'\Sigma t}{2}.
\end{aligned}
$$

□

Corollary 1 *Suppose that $X \sim \mathrm{N}(\mu, \Sigma)$. Then*

$$
\begin{aligned}
\mathrm{E}(X) &= \mu, \\
\mathrm{V}(X) &= \Sigma.
\end{aligned}
$$

Proof. Exercise.

Corollary 2 *Suppose that $X \sim \mathrm{N}(\mu, \Sigma)$. Then*

$$AX + b \sim \mathrm{N}(A\mu + b, A\Sigma A').$$

Proof. Exercise.
Remark: In particular, for $i = 1, \ldots, n$, $X_i \sim \mathrm{N}\left(\mu_i, \sigma_i^2\right)$.

3 Independence

Theorem 2 *Suppose that $X \sim \mathrm{N}(\mu, \Sigma)$. Then $X_1, \ldots, X_n$ are independent iff they are uncorrelated.*

Proof. "$\Longrightarrow$" Standard. "$\Longleftarrow$" Since Σ is diagonal,

$$
\begin{aligned}
\det(\Sigma) &= \sigma_1^2 \cdots \sigma_n^2, \\
\Sigma^{-1} &= \begin{bmatrix} \sigma_1^{-2} & & 0 \\ & \ddots & \\ 0 & & \sigma_n^{-2} \end{bmatrix}.
\end{aligned}
$$

So $\forall x \in \Re^n$,

$$
\begin{aligned}
f(x) &:= (2\pi)^{-n/2}\det(\Sigma)^{-1/2}\exp\left(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right) \\
&= (2\pi)^{-n/2}\left(\sigma_1^2 \cdots \sigma_n^2\right)^{-1/2}\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu_i)^2}{\sigma_i^2}\right) \\
&= \prod_{i=1}^{n}(2\pi)^{-1/2}\left(\sigma_i^2\right)^{-1/2}\exp\left(-\frac{(x_i-\mu_i)^2}{2\sigma_i^2}\right) \\
&= f_1(x_1)\cdots f_n(x_n).
\end{aligned}
$$

□

4 Conditional Distribution

Let $X := (X_1, X_2)'$.

Theorem 3 *Suppose that*

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right).$$

Then

$$X_2|X_1 \sim \mathrm{N}\left(\mu_{2|1}, \Sigma_{22|1}\right),$$

where

$$\begin{aligned} \mu_{2|1} &:= \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(X_1 - \mu_1), \\ \Sigma_{22|1} &:= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}. \end{aligned}$$

Proof. We have

$$\begin{aligned} \det(\Sigma) &= \det(\Sigma_{11}) \det\left(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right) \\ &= \det(\Sigma_{11}) \det\left(\Sigma_{22|1}\right). \end{aligned}$$

So $\forall x \in \Re^n$,

$$\begin{aligned} f_{2|1}(x_2|x_1) &:= \frac{f(x)}{f_1(x_1)} \\ &= \frac{(2\pi)^{-n/2}\det(\Sigma)^{-1/2}\exp\left(-(x-\mu)'\Sigma^{-1}(x-\mu)/2\right)}{(2\pi)^{-n_1/2}\det(\Sigma_{11})^{-1/2}\exp\left(-(x_1-\mu_1)'\Sigma_{11}^{-1}(x_1-\mu_1)/2\right)} \\ &= (2\pi)^{-n_2/2}\det(\Sigma_{22|1})^{-1/2} \\ &\quad \exp\left(-\frac{1}{2}\left[(x-\mu)'\Sigma^{-1}(x-\mu) - (x_1-\mu_1)'\Sigma_{11}^{-1}(x_1-\mu_1)\right]\right). \end{aligned}$$

WTS

$$\begin{aligned} &(x-\mu)'\Sigma^{-1}(x-\mu) - (x_1-\mu_1)'\Sigma_{11}^{-1}(x_1-\mu_1) \\ &\quad = \left[x_2 - \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)\right]'\Sigma_{22|1}^{-1}\left[x_2 - \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)\right]. \end{aligned}$$

We have

$$\Sigma^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$\begin{aligned} B_{11} &:= \Sigma_{11}^{-1} + \Sigma_{11}^{-1}\Sigma_{12}\left(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)^{-1}\Sigma_{21}\Sigma_{11}^{-1} \\ &= \Sigma_{11}^{-1} + \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22|1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}, \\ B_{12} &:= -\Sigma_{11}^{-1}\Sigma_{12}\left(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)^{-1} \\ &= -\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22|1}^{-1}, \\ B_{21} &:= -\left(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)^{-1}\Sigma_{21}\Sigma_{11}^{-1} \\ &= -\Sigma_{22|1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}, \\ B_{22} &:= \Sigma_{22|1}^{-1}. \end{aligned}$$

So

$$
\begin{aligned}
&(x-\mu)'\Sigma^{-1}(x-\mu)-(x_1-\mu_1)'\Sigma_{11}^{-1}(x_1-\mu_1) \\
&\quad = (x_1-\mu_1)'\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22|1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)-2(x_1-\mu_1)'\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22|1}^{-1}(x_2-\mu_2) \\
&\qquad +(x_2-\mu_2)'\Sigma_{22|1}^{-1}(x_2-\mu_2) \\
&\quad = (x_2-\mu_2)'\Sigma_{22|1}^{-1}(x_2-\mu_2)-2(x_2-\mu_2)'\Sigma_{22|1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1) \\
&\qquad \left[\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)\right]'\Sigma_{22|1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1) \\
&\quad = \left[(x_2-\mu_2)-\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)\right]'\Sigma_{22|1}^{-1}\left[(x_2-\mu_2)-\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1)\right].
\end{aligned}
$$

□

A Partitioned Matrices

A.1 Partitioned Matrices

Let

$$
A := \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{bmatrix},
$$

where for $i = 1, \dots, m$, for $j = 1, \dots, n$, A_{ij} is $m_i \times n_j$.

Definition 3 *A is* upper (lower) block triangular *if*

1. *the diagonal submatrices are square,*
2. *the lower (upper) off-diagonal submatrices are* 0*s.*

Remark: A block triangular matrix is square.

Definition 4 *A is* block diagonal *if*

1. *the diagonal submatrices are square,*
2. *the off-diagonal submatrices are* 0*s.*

A.2 Matrix Algebra

Let

$$
\begin{aligned}
A &:= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\
B &:= \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.
\end{aligned}
$$

Then

$$
\begin{aligned}
A+B &= \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \end{bmatrix}, \\
AB &= \begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix},
\end{aligned}
$$

provided that they are conformable.

A.3 Inverses

Let A be a square matrix s.th.

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square.

Theorem 4 *Suppose that A and A_{11} are invertible. Then*

$$A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$\begin{aligned}
B_{11} &= A_{11}^{-1} + A_{11}^{-1} A_{12} \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} A_{21} A_{11}^{-1}, \\
B_{12} &= -A_{11}^{-1} A_{12} \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1}, \\
B_{21} &= -\left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} A_{21} A_{11}^{-1}, \\
B_{22} &= \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
A_{11} B_{11} + A_{12} B_{21} &= I, \\
A_{11} B_{12} + A_{12} B_{22} &= 0, \\
A_{21} B_{11} + A_{22} B_{21} &= 0, \\
A_{21} B_{12} + A_{22} B_{22} &= I.
\end{aligned}$$

Since A_{11} is invertible,

$$\begin{aligned}
B_{11} &= A_{11}^{-1}(I - A_{12} B_{21}), \\
B_{12} &= -A_{11}^{-1} A_{12} B_{22}.
\end{aligned}$$

Plugging these into the third and forth equations,

$$\begin{aligned}
A_{21} A_{11}^{-1}(I - A_{12} B_{21}) + A_{22} B_{21} &= 0, \\
-A_{21} A_{11}^{-1} A_{12} B_{22} + A_{22} B_{22} &= I,
\end{aligned}$$

or

$$\begin{aligned}
B_{21} &= -\left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} A_{21} A_{11}^{-1}, \\
B_{22} &= \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1}.
\end{aligned}$$

Plugging these back,

$$\begin{aligned}
B_{11} &= A_{11}^{-1} \left[I + A_{12} \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} A_{21} A_{11}^{-1}\right] \\
&= A_{11}^{-1} + A_{11}^{-1} A_{12} \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} A_{21} A_{11}^{-1}, \\
B_{12} &= -A_{11}^{-1} A_{12} \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1}.
\end{aligned}$$

□

Remark: A similar result holds when A_{22} is invertible.

A.4 Determinants

Lemma 1 *Suppose that A is block triangular. Then*

$$\det(A) = \det(A_{11})\det(A_{22}).$$

Remark: This follows from Laplace's expansion theorem.

Theorem 5 *Suppose that A_{11} is invertible. Then*

$$\det(A) = \det(A_{11})\det\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right),$$

Proof. Let

$$B := \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}.$$

By the previous lemma.

$$\det(B) = 1.$$

So

$$\begin{aligned}
\det(A) &= \det(A)\det(B) \\
&= \det(AB) \\
&= \det\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}\right) \\
&= \det(A_{11})\det\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right).
\end{aligned}$$

□

Remark: A similar result holds when A_{22} is invertible.