A Reconsideration of the Dynamic Laffer Curve in a Two-Sector Model of Endogenous Growth*

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I. Introduction

Recent studies by Ireland (1994) and Pecorino (1995) have reexamined the Laffer curve, taking account of the growth effect. In these new studies, in addition to the disincentive effect on static labor supply decisions, a tax policy can affect the long run growth rate of an economy. More specifically, a lower growth rate in response to a higher tax rate may decrease the future tax base and therefore future tax revenues.

To date, studies such as those by Pecorino (1995) and Ireland (1994) have restricted their analyses to steady states. However, the omission of analysis of transition paths has important consequences for evaluating present values, because the interest rate can change during a transition. Pecorino (1995), for example, does not make any attempt to capture this effect. In addition, as the tax rate becomes higher and higher, the speed of convergence toward the steady state decelerates, the economy stays longer on the transition path, and the transition path becomes more important. The discounting of future tax revenues for Laffer-curve analysis adds more significance to the short run.

Different ways of evaluating streams of tax revenues in the existing literature have created large differences in their results: in Ireland (1994) the tax rate at the turning point of the Laffer curve is 15% (this means that cutting the tax rate from the current rate is self-financing in his model), while in Pecorino (1995) the present value of tax revenues is maximized at 64% (i.e., the turning point of his Laffer curve is 64%). Given these disparities and complexities, we need to reevaluate results from previous studies, comparing them within a unified endogenous growth framework, using a unified measure.

This paper first shows that, in contrast with the Ramsey-Cass-Koopmans model, studying the maximum point of the present value of tax revenues may not be useful, and an alternative measure of comparing tax revenues is required in an endogenous growth model. In terms of the Laffer curve, the present value of tax revenues can be differently interpreted as follows: (a) the initial debt which can be financed; (b) the government expenditure \( G \) that can be financed, where government expenditure per efficiency units is constant, but \( G \) is growing at the base year growth rate; (c) the constant \( G \) that can be financed; and (d) the constant \( G/Y \) that can be financed. It is then shown that comparing tax revenues in terms of (a) or (d) is not useful from a viewpoint of an endogenous growth, and that comparing tax revenues in different tax regimes in terms of (b) or (c) is more appropriate. Among the previous studies, Ireland (1994) implicitly considers (b), while Pecorino (1995) might be interpreted as trying to partially capture the concept (c) by using a constant discount rate.

This paper then analyzes the relationship between tax rates and the stream of government expenditures that can be financed, in terms of (b) and (c). From this point of view, a turning point of the Laffer curve occurs at the tax rate which maximizes the present value of a constant stream of government expenditures

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1Stokey and Rebelo (1995) analyzed only the relationship between tax rates and the share of tax revenue in total income, while Ireland (1994) and Pecorino (1995) analyzed tax revenues in terms of the present value.

2For studies on the traditional static Laffer curve, see, for example, Fullerton (1982).

3This effect sharply contrasts with that of the traditional Ramsey-Cass-Koopmans type exogenous growth model where the long run growth rate of the economy is determined by the exogenous rate of technical progress, and therefore tax policies cannot affect the long run growth rate.

4Ireland (1994) uses a simple one-sector AK model, and therefore the economy is always on the balanced growth path and has no transitional dynamics. Pecorino (1995) adopts a two-sector endogenous growth model, but restricts his analysis to comparisons of tax revenues across different countries in steady states.

5However, his measure suffers from the price evaluation problem discussed in the next section.

6His measure of the Laffer curve is the present value of tax revenues itself, and thus it may be roughly interpreted as (a). However, his attempt to use a distortion-free constant rate of return as a discount rate may be interpreted as his trying to capture other concepts such as (c). See the next section in more detail.
that can be financed by the present value of tax revenues. Under the parameter set of Pecorino (1995), assuming the initial wage tax rate is 40% as in Lucas (1990), such a tax rate is 46% in terms of (b), and 53% in terms of (c). This means that a tax-cut is not self-financing, unlike the result of Ireland (1994), since the growth rate effect is smaller than in his one-sector AK model. Thus, the economy is on the upward-sloping part of the Laffer curve. At the same time, these rates are smaller than Pecorino (1995) finds in making comparisons of the present value of tax revenues across different countries in steady states. This result occurs because we are able to capture the effects of the change in the interest rate during the transition path, and evaluate tax revenues in a single economy. However, these results critically depend on parameter values such as the elasticity of intertemporal substitution and that of labor supply. If these elasticities are smaller, the resulting turning point occurs at an even higher tax rate. On the other hand, if the elasticity of intertemporal substitution is near or greater than 1, or if the labor supply is fairly elastic, the above results can be overturned. That is, a tax-cut can be self-financing. Therefore we can obtain a welfare gain by lowering the tax rate, while still maintaining the government’s present value budget constraint even in a two-sector endogenous growth model where a human capital accumulation is lightly taxed; thus the growth rate effect is relatively small.7

This paper also shows that conflicting results of previous studies mainly come from different concepts of the Laffer curve and different key parameter values. By evaluating the results of these studies with a unified measure in a unified framework, we find that differences are not as great as Pecorino (1995) argues. In fact, the result is in between that of Ireland (1994) and of Pecorino (1995): under the parameter set in the sensitivity analysis of Pecorino, Ireland’s conclusion can hold; a turning point of the Laffer curve is lower than that of Pecorino, but higher than that of Ireland under Pecorino’s basic parameter set. Under empirically more relevant parameter values, a tax-cut is not self-financing, and the turning point of the Laffer curve is relatively high. Even if a tax-cut can be self-financing, however, the resulting welfare gain by lowering the tax rate to the turning point is much smaller than Ireland (1994) finds. By taking account of the transition path, with some factors weakening the growth rate effect, this paper thus obtains different results from those of previous studies.

For the purposes of this paper, tax revenues have been analyzed under the following condition: following an unexpected, abrupt, and permanent change in flat tax rates, using an endogenous growth framework like those in the existing literature, and taking into account equilibrium transition paths. The same tax structure as in Pecorino (1995) is assumed for comparisons: a human capital accumulation is less highly taxed.8 Because of the nonlinearity of the system, we numerically analyze this problem.9 The procedure is briefly as follows. First, the system of nonlinear difference equations at the steady state is linearized, and eigenvalues and eigenvectors are computed. Next, saddlepoint stability is checked from the eigenvalues, and the transition path is solved nonlinearly by shooting backwards from the new steady state in the direction of the eigenvector specifying the stable branch. This procedure is the extended backward shooting method, combining with Laitner (1995)’s linearization method. Finally, tax revenues are evaluated in terms of a unified measure.

The remainder of this paper is organized as follows. Section II defines the Laffer curve, compares the results of the previous studies in a simple AK growth model, and explains their relationships and intuitive meanings. Section III briefly describes the model, and characterizes the steady state and transitional dynamics. All of the eigenvalues at the steady states are computed. The economy is shown to be saddle-path stable in different equilibria under plausible parameter values. Section IV reports simulation results, while section V conducts a sensitivity analysis with respect to parameter values. Section VI concludes.

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7The disincentive effect of taxation on labor supply was emphasized by the previous studies on the traditional static Laffer curve. The same effect has another channel through which distorted labor supply decisions has the negative effect on the human capital accumulation, and therefore on the growth rate in this model. The tax treatment of depreciation also has a large effect on the growth rate. A tax-cut can be self-financing or not, depending on the assumed tax treatments of depreciation allowances even in a one-sector AK model.

8It is because, to some extent, “the human capital input into human capital accumulation reflects foregone earnings (which are untaxed)”, and the physical capital in the human capital production sector reflects “the nonprofit nature of higher education”. See Pecorino (1995), p.533.

II. Definition of the Laffer Curve

The Laffer curve in the traditional Ramsey-Cass-Koopmans model and that in an endogenous growth model should be carefully distinguished.\(^{10}\) In the traditional model, the economy’s long run growth rate of per-capita variables is given by the exogenous rate of technical progress. Since the long run growth rate is constant and thus tax policies cannot affect it, we can finance a greater amount of the government expenditure per efficiency units by raising the present value of tax revenues. Assume for the moment that both the rate of technical progress and that of population growth are 0.\(^{11}\) Chamley (1985) shows that the increase in the present value of tax revenues by raising the tax rate from \(\tau\) to \(\tau + \Delta \tau\) can be approximated (up to the first order) as follows:\(^{12}\)

\[
\Delta PVR \approx \frac{WR}{\rho},
\]

where \(WR = \frac{\kappa}{\rho + \kappa}TR_0 + \frac{\kappa}{\rho + \kappa}TR' - TR = \frac{\rho}{\rho + \kappa}(TR_0 - TR) + \frac{\kappa}{\rho + \kappa}(TR' - TR), \rho\) is the rate of time preference, \(\kappa\) the convergence speed, \(TR\) the tax revenue at the old steady state, \(TR'\) the tax revenue at the new steady state, and \(TR_0\) the tax revenue at time 0 when the tax rate is changed. Appendix A shows that in an exogenous labor supply model, for a given \(\tau, TR\) is constant, \(TR_0\) is linearly increasing in \(\Delta \tau\), and \(TR'\) is strictly concave, and increasing at first but decreasing beyond some tax rate \(\tau + \Delta \tau\), and therefore, \(WR\) increases at first, but may decrease beyond some tax rate.\(^{13}\) This then is the Laffer curve of the Ramsey-Cass-Koopmans model. Now assume that we set a constant increment of government expenditure, \(\Delta g\), equal to \(WR\). Then, the maximum point of \(\Delta g\) coincides with that of \(\Delta PVR\) since the increase in the present value of government expenditures, \(\Delta PVG\), can be approximated by\(^{15}\)

\[
\Delta PVG \approx \int_0^\infty \Delta g e^{-\rho t} dt = \frac{\Delta g}{\rho}.
\]

From the above argument, the income tax rate \(\tau\) yielding the maximal present value of tax revenues roughly coincides with the \(\tau\) yielding the maximal constant level of \(g\) that can be financed. Therefore, in this case, it is meaningful to study the size of the present value of tax revenues. In an endogenous growth model, however, these two maximum points do not generally coincide. This suggests that checking the maximum point of the present value of tax revenues is not useful, and an alternative measure for comparing tax revenues is required in an endogenous growth framework.

In an endogenous growth model, tax policies can affect the long run growth rate. Thus, even if the present value of tax revenues increases by raising the tax rate, we may not be able to finance the original (before tax rate change) stream of government expenditures since the long run growth rate may become too low.\(^{16}\) In this case, it is not meaningful to compare the size of the present value of tax revenues under different tax policies. In interpreting the size of the present value of tax revenues, some possible bases for comparison are as follows: (a) the initial debt which can be financed; (b) the government expenditure \(G\) that can be financed, where government expenditure per efficiency units, \(g\), is constant, but \(G\) is growing at the base year growth rate; (c) the constant \(G\) that can be financed; (d) the constant \(G/Y\) that can be financed; and (e) the scale of an arbitrary pattern of \(G\) and initial debt that can be financed.

There are two approaches to the Laffer curve in a framework of an endogenous growth: those of Ireland (1994) and Pecorino (1995). For the explanation of their approaches in a single framework, consider the following AK growth model. The production function is given by \(Y = F(K) = AK\), where \(Y\) is an output, \(A\) is a constant, and \(K\) is a composite of physical and human capital. Assuming the constant relative risk aversion utility, the economy’s growth rate \(\nu\) is endogenously determined by \(\nu = \frac{\gamma}{\sigma} \frac{\gamma - \rho}{\sigma}g\), where \(c\) is a consumption per capita, \(\sigma\) the inverse of the elasticity of intertemporal substitution, \(\rho\) the rate of time

\(^{10}\)Only in this section, the model is described by the continuous-time model. It is because continuous-time is better for obtaining economic intuition and for analytical purposes even though discrete-time is best for computation. From section III, the model is described by the discrete-time model mainly for computational purposes. The continuous-time version of the model in this section is considered to be the continuous-time limit of the discrete-time problem in the following sections.

\(^{11}\)This simplifies the following argument, but does not change the result, since both rates are exogenously fixed and tax policies cannot affect them.


\(^{13}\)Although this argument is based on the linear approximation, the similar argument may be possible by numerical simulations.

\(^{14}\)The government expenditure is assumed to be a lump-sum transfer. If the rate of technical progress and that of population growth are positive, \(\Delta g\) is a constant increment of government expenditure per efficiency units of labor.

\(^{15}\)Apply the theorem of Chamley (1985).

\(^{16}\)As shown below, even if the growth rate is negative, the present value of tax revenues can increase since the interest rate becomes low enough to offset the negative growth rate effect.
preference, and \( \gamma \) the after-tax interest rate net of depreciation. Then, \( \rho + \sigma \nu = \gamma. \) Assuming no tax deductibility of capital depreciation expense, \( \gamma = (1-\tau)A - \delta \) holds, where \( \tau \) is the income tax rate and \( \delta \) the depreciation rate. This economy is always on the steady state, and has no transitional dynamics. Tax revenue each period is \( TR_t = \tau AK_0 e^{\rho t}, \) where \( K_0 \) is the initial capital stock. Initially, government budget is balanced each period, i.e., \( TR_t = G_t = g e^{\rho t}, \) where \( g \) is a constant, and the initial government debt is assumed to be 0. Government expenditure is assumed to be a lump-sum transfer. The present value of tax revenues, \( PVR, \) is

\[
PVR = \int_0^\infty TR_t e^{-\gamma t} dt = \tau AK_0 \int_0^\infty e^{(\nu - \gamma) t} dt = \frac{\tau AK_0}{\gamma - \nu} = \frac{AK_0}{\rho + (\sigma - 1)\nu}.\]

Assume for simplicity that \( \sigma = 1. \) Then \( PVR \) is a linear function of the tax rate. In this case, \( PVR \) is always increasing in \( \tau, \) even if the economy’s growth rate becomes negative. Therefore, we may not be able to finance the original stream of government expenditures \( \{g e^{\nu t}\} \) even if the present value of tax revenues increases by raising the tax rate. The size of \( PVR \) itself may be roughly interpreted as the initial national debt which can be financed, (a).\(^{19}\)

In order to study the Laffer curve without analyzing the transition path, Pecorino (1995) “makes comparisons of the present value of revenue collections across countries with differing level of taxation.”\(^{20}\) Thus, it is necessary for him to use a consistent discount rate for comparisons of the present value across different economies, and therefore he uses as \( \gamma \) a constant steady state interest rate net of depreciation in a tax-free economy. That is, Pecorino (1995) assumes that \( \gamma \) is an interest rate net of depreciation in a tax-free economy \( \gamma_d. \) \( PVR \) thus takes the form of:

\[
PVR^P = \frac{\tau AK_0}{\gamma_d - \nu}.\]

The relationship between tax rates and \( PVR^P \) is the Laffer curve of Pecorino (1995). Since \( \nu \) is a decreasing function of \( \tau \) and \( \gamma_d \) is fixed, \( PVR^P \) may decrease beyond some tax rate if the growth rate effect is strong compared to the direct revenue-increasing effect of the numerator. In this simple model, however, the Laffer curve has no turning point. This can be easily seen by transforming (4) as follows:

\[
PVR^P = \frac{\tau AK_0}{\gamma_d - \nu} = \frac{\tau AK_0}{\tau A - \rho} = \frac{AK_0}{A + \rho/\tau}.\]

Clearly, \( PVR^P \) is always increasing in \( \tau. \) In the more general model of Pecorino (1995), the increase in denominator outweighs the increase in numerator as the tax rate becomes high.\(^{21}\) This is why Pecorino (1995) obtains a turning point of his Laffer curve across different economies.\(^{22}\)

In comparing the present values of tax revenues in a single economy with different levels of taxation, however, we should include the effect of the change in the interest rate in calculating the present value. Then, as explained above, the present value of tax revenues always increases, and therefore, there is no turning point.\(^{23}\) In spite of this, we may not be able to finance the original stream of government expenditures by

\(^{17}\)From this, \( \Delta \nu = \Delta \gamma/\sigma. \) Thus, the smaller \( \sigma, \) the larger the growth rate effect.

\(^{18}\)Also when \( \sigma > 1, PVR \) is always increasing since \( \nu \) is decreasing in \( \tau. \)

\(^{19}\)If the initial debt is not 0 and tax revenues are used only for the repayment of the debt and the interest payments, then \( TR_t = \gamma d_t - d_t, \) where \( d_t \) is a government debt. Thus, in this case, \( d_0 = \int_0^\infty TR_t e^{-\gamma t} dt. \)

\(^{20}\)His model is a more general two sector endogenous growth model, and therefore the economy is on the transition path after the change in the tax rate. However, in order to restrict his analysis on the steady state, his way to envision his experiment is “to consider many economies, each experiencing a different level of taxation and each on its balanced growth path. At the instant the calculation for the experiment is performed, each economy is constrained to have the same potential GNP” (Pecorino (1995), p.532).

\(^{21}\)Define \( PVR^P(\tau) = \frac{\tau AK_0}{\gamma_d - \nu(\tau)}, \) where \( \gamma_d = \rho + \nu(0). \) Then, \( PVR^P(\tau) = \frac{AK_0}{\gamma_d - \nu(\tau) + \nu'(\tau)} \). In order for a turning point to exist, the inequality \( \gamma_d - \nu(\tau) + \nu'(\tau) < 0 \) must hold beyond some \( \tau. \) In the case of a simple AK model above, this inequality can never hold because of the linearity of the growth rate function \( \nu(\tau). \) If, for example, \( \nu(\tau) \) is strictly concave (see figure 2A), this inequality can hold even if the growth rate effect is relatively small for a low \( \tau. \) In addition, if the initial tax base \( AK_0 \) shrinks as the tax rate becomes high, we have a more chance of obtaining a turning point. This happens in the case of Pecorino (1995), since he assumes an endogenous labor supply. See Appendix B in detail and for the more general case.

\(^{22}\)Pecorino (1995) adopts the two sector endogenous growth model to capture the weaker growth rate effect than in Ireland (1994). In order to compare the present value of tax revenues in one economy, the whole movement of the interest rate stream after the change in the tax rate must be taken into account, and therefore the transition path must be analyzed. Since he does not make any attempts to analyze the transition path, he limits his analysis to comparisons of the present value of tax revenues across different economies.

\(^{23}\)In (3), \( PVR \) is expressed as \( PVR = \frac{AK_0}{\rho} \tau. \)
raising the tax rate. This outcome means that from a viewpoint of comparing the tax revenues in a single economy, it is not meaningful to compare the sizes of the present values of tax revenues themselves.

In contrast, Ireland (1994) focuses on financing the original stream of government expenditures in a single economy but with different levels of taxation. His experiment is as follows. Suppose that the economy is on the steady state at the tax rate of $\tau^0$. Then, $\gamma^0 = (1 - \tau^0)A - \delta$, and $\nu^0 = \gamma^0 - \rho$. The government budget is balanced for each period. He asks if we can cut the tax rate from $\tau^0$ to $\tau^1$ ($\tau^1 < \tau^0$), and keep the original stream of government expenditures $\{g e^{\nu^1 t}\}$. Thus, he implicitly considers (b). The $PVR$ and the present value of the original stream of government expenditures, $PVG$, can be expressed as

$$\begin{align*}
(5) & \quad PVR^* = \int_0^\infty \tau^1 AK_0 e^{\nu^1 t} e^{-\gamma^1 t} dt = \frac{\tau^1 AK_0}{\gamma^1 - \nu^1} = \frac{AK_0}{\rho} \gamma^1, \\
(6) & \quad PVG^* = \int_0^\infty \tau^0 AK_0 e^{\nu^0 t} e^{-\gamma^1 t} dt = \frac{\tau^0 AK_0}{\gamma^1 - \nu^0} = \frac{\tau^0 AK_0}{(\gamma^1 - \nu^1) + (\nu^1 - \nu^0)} = \frac{\tau^0 AK_0}{\rho + (\nu^1 - \nu^0)},
\end{align*}$$

where $\gamma^1 = (1 - \tau^1)A - \delta$, and $\nu^1 = \gamma^1 - \rho$. When $\tau^1 < \tau^0$, then $\nu^1 > \nu^0$. Therefore, if the growth rate effect is large (i.e., $\nu^1$ is large enough compared to $\nu^0$), $PVR^*$ may be larger than $PVG^*$ for a lower tax rate $\tau^1$. This means that we can finance the original stream of government expenditures by lowering the tax rate.\(^{25}\) However, if the growth rate effect is very small ($\nu^1 - \nu^0 \approx 0$),\(^{26}\) this may not be true. His conclusion crucially depends on the size of the growth rate effect.

Under the assumption of no tax deductibility of capital depreciation expense as in Ireland (1994), $PVG^*$ is

$$\begin{equation}
(7) \quad PVG^* = \frac{\tau^0 AK_0}{\rho + (A - \delta)(\tau^0 - \tau^1)}.
\end{equation}$$

Under the parameter set of King and Rebelo (1990),\(^{27}\) he can find that a lower tax rate can finance the original stream of government expenditures. However, if full tax deductibility of capital depreciation expense is assumed, $\gamma = (A - \delta)(1 - \tau)$ holds. Then,

$$\begin{equation}
(7') \quad PVG^* = \frac{\tau^0 (A - \delta)K_0}{\rho + (A - \delta)(\tau^0 - \tau^1)}.
\end{equation}$$

As a result, the growth rate effect is weakened.\(^{28}\) Then, we cannot finance the original stream of government expenditures by lowering the tax rate if $A - \delta$ is small.\(^{29}\)

In a more realistic two sector endogenous growth model, even if full tax deductibility of depreciation expense is not assumed, there exist other factors which weaken the growth rate effect. Stokey and Rebelo (1995) show that “if human capital’s share is large in all sectors, if the sector producing human capital is lightly taxed, and if long-run labor supply is fairly inelastic, then taxing returns in the sectors producing consumption goods and physical capital does not have large growth effects”.\(^{30}\) Under such conditions, we can no longer finance the original stream of government expenditures by lowering the tax rate.\(^{31}\)

The government budget deficit each period is assumed to be financed by issuing the government debt. By debt neutrality, this government debt can be replaced with the lump-sum tax. Thus, his experiment can be also expressed as follows. Collect the tax revenue by the income tax, and return it as a lump-sum transfer. If this tax revenue is short of the required original level of government expenditure, then collect the rest of the tax revenue by the lump-sum tax, and return it as a lump-sum transfer. If the revenue collected by the income tax exceeds the required level of government expenditure, then the similar argument also applies. By the nature of the lump-sum tax, this economy is equivalent to the economy which collects the tax revenue only by the income tax, and returns it in a lump-sum manner. Therefore, we do not have to explicitly consider the government debt nor the effects without government’s budget balance for each period on the economic activity.

In this case, we cannot finance the original stream of government expenditures by raising the tax rate.\(^{26}\) For example, as explained below, consider the case where $\nu^1 - \nu^0 = (A - \delta)(\tau^0 - \tau^1)$ and $A - \delta$ is very small.

\(^{25}\)This effect is emphasized by Stokey and Rebelo (1995). The real economy lies in between these two extreme cases since depreciation of physical capital is at least partially deductible in the US.

\(^{26}\)Actually, Pecorino (1995) incorporates these factors and obtains the smaller growth rate effect by using the two sector endogenous growth model, and shows that a turning point of the Laffer curve is relatively high, compared to that of Ireland (1994) from a viewpoint of comparing present values across countries with different levels of taxation. Our interest is, however, in comparing tax revenues in a single economy, and thus is different from his. In addition, his method may not be appropriate in our context, as discussed above.

\(^{30}\)This point will be clear in (14) below.
In addition to the above questions, Ireland (1994) asks what tax rate yields the maximum excess of present value of tax revenues over that of the original stream of government expenditures.\(^{32}\) Change the tax rate from \(\tau^0\) to \(\tau^1\) (\(\tau^0 \neq \tau^1\)), then the government’s present value budget constraint is

\[
\int_0^\infty (g + \Delta g) e^{-\gamma^1 t} dt = \int_0^\infty \tau^1 AK_0 e^{\nu^1 t} e^{-\gamma^1 t} dt,
\]

where \(\Delta g\) is a constant increment of \(g\). This can be transformed as

\[
\frac{g}{\gamma^1 - \nu^0} + \frac{\Delta g}{\gamma^1 - \nu^0} = \frac{\tau^1 AK_0}{\gamma^1 - \nu^1}, \Leftrightarrow \frac{\Delta g}{\rho + (\nu^1 - \nu^0)} = \frac{\tau^1 AK_0}{\rho}. \tag{9}
\]

Therefore, the present value of tax revenues minus that of the original stream of government expenditures is

\[
\int_0^\infty \Delta g e^{\nu^0 t} e^{-\gamma^1 t} dt = \int_0^\infty \tau^1 AK_0 e^{\nu^1 t} e^{-\gamma^1 t} dt - \int_0^\infty ge^{\nu^0 t} e^{-\gamma^1 t} dt, \Leftrightarrow \frac{\Delta g}{\gamma^1 - \nu^0} = \frac{\tau^1 AK_0}{\gamma^1 - \nu^1} - \frac{g}{\gamma^1 - \nu^0}. \tag{10}
\]

Ireland (1994) calculates the tax rate which maximizes \(\frac{\Delta g}{\gamma^1 - \nu^0}\). The relationship between the tax rate \(\tau^1\) and \(\frac{\Delta g}{\gamma^1 - \nu^0}\) is thus the Laffer curve of Ireland (1994).\(^{33}\)

However, the maximum point of \(\frac{\Delta g}{\gamma^1 - \nu^0}\) may not coincide with that of \(\Delta g\). The maximum point of \(\frac{\Delta g}{\gamma^1 - \nu^0}\) is not independent of the prices (interest rates) at which the value is measured. Therefore, in this paper, we define the Laffer curve as the relationship between tax rates and \(\Delta g\) in order to eliminate this problem. A turning point of the Laffer curve is given by the maximum point of \(\Delta g\) with respect to \(\tau\).

Then, what’s the tax rate that maximizes \(\Delta g\)? Since \(g = \tau^0 AK_0\), \(\Delta g\) can be expressed as

\[
\Delta g = AK_0 \left[ \left( \frac{\rho + (\nu^1 - \nu^0)}{\rho} \right) \tau^1 - \tau^0 \right] = AK_0 \left[ \left( 1 + \frac{(\nu^1 - \nu^0)}{\rho} \right) \tau^1 - \tau^0 \right]. \tag{11}
\]

If \(\nu^1 - \nu^0 = A(\tau^0 - \tau^1)\) as in Ireland (1994), then

\[
\Delta g = AK_0 \left[ \left( 1 + \frac{(\nu^1 - \nu^0)}{\rho} \right) \tau^1 - \tau^0 \right] = AK_0 \left[ \left( 1 + \frac{A(\tau^0 - \tau^1)}{\rho} \right) \tau^1 - \tau^0 \right] = -AK_0 \frac{A}{\rho} (\tau^1 - \tau^0)(\tau^1 - \frac{\rho}{A} \tau^0) \equiv \Delta g(\tau^1). \tag{12}
\]

\(\Delta g\) is a quadratic function of the tax rate. Maximizing \(\Delta g(\tau^1)\) with respect to \(\tau^1\), we obtain

\[
\tau^{1*} = \frac{\rho}{2A} + \frac{\tau^0}{2}. \tag{13}
\]

Under the parameter set of King and Rebelo (1990), \(\tau^{1*} = 0.136.\)\(^{34}\) Thus, a tax-cut is self-financing since \(\tau^0\) is assumed to be 20%. If \(\nu^1 - \nu^0 = (A - \delta)(\tau^0 - \tau^1)\) as in the case of full tax deductibility of depreciation expense, then\(^{35}\)

\[
\tau^{1*} = \frac{\rho}{2(A - \delta)} + \frac{\tau^0}{2}. \tag{14}
\]

\(^{32}\)This can be considered as trying to interpret the present value of tax revenues as \(b\).

\(^{33}\)This is not exactly correct. In a discrete time model, Ireland (1994) calculates the tax rate that maximizes \(\frac{\Delta PV}{\tau^1 - \tau^0}\), and his Laffer curve is the relationship between tax rates and this expression. This can be interpreted as trying to partially eliminate the price evaluation problem discussed immediately below. However, this measure still suffers from the same problem. In fact, we can obtain the same result as his by maximizing \(\frac{\Delta g}{\gamma^1 - \nu^0}\). See below.

\(^{34}\)Ireland (1994) also uses the parameter set of King and Rebelo (1990), and obtains the different result \(\tau^{1*} = 0.15\). It is because of the price evaluation problem discussed above. See (16).

\(^{35}\)In this case, \(\Delta g(\tau^1) = -(A - \delta)K_0 \frac{A}{\rho} (\tau^1 - \tau^0)(\tau^1 - \frac{\rho}{A} \tau^0)\).
Under the parameter set of King and Rebelo (1990),\(^{36}\) \(\tau^{1*} = 0.25\). Therefore, under the assumption of full tax deductibility of depreciation expense, a tax-cut is no longer self-financing.\(^{37}\) As discussed above, the maximum point of \(\Delta g\) may not coincide with that of \(\frac{\Delta g}{\gamma_1 - \nu^2}\). If \(\nu^1 - \nu^0 = A(\tau^0 - \tau^1)\), then

\[
\frac{\Delta g}{\gamma_1 - \nu^2} = \frac{\Delta g}{\rho + (\nu^1 - \nu^0)} = \frac{\Delta g}{\rho + A(\tau^0 - \tau^1)} = AK_0 \left[ \frac{\tau^1}{\rho} - \frac{\tau^0}{\rho + A(\tau^0 - \tau^1)} \right].
\]

Maximize this with respect to \(\tau^1\), then we obtain\(^{38}\)

\[
\tau^{1*} = \frac{p}{A} + \tau^0 - \sqrt{\frac{p\tau^0}{A}} = \left( \sqrt{\frac{p}{A} - \sqrt{\tau^0}} \right)^2 + \sqrt{\frac{p\tau^0}{A}}.
\]

This is clearly different from (13). Under the parameter set of King and Rebelo (1990), \(\tau^{1*} = 0.152\). This is the result that Ireland (1994) obtains.

In the interpretation (c), equation (8) is modified as

\[
\int_0^\infty G e^{-\gamma_1 t} dt = \int_0^\infty \tau^1 AK_0 e^{\nu_1 t} e^{-\gamma_1 t} dt.
\]

Then,\(^{39}\)

\[
G = \frac{\int_0^\infty \tau^1 AK_0 e^{\nu_1 t} e^{-\gamma_1 t} dt}{\int_0^\infty e^{-\gamma_1 t} dt} = \frac{AK_0}{\rho} \tau^1 \gamma_1 = \frac{AK_0}{\rho} \tau^1(A(1 - \tau^1) - \delta).
\]

This is maximized at

\[
\tau^{1*} = \frac{1}{2} - \frac{\delta}{2A}.
\]

Under the parameter set of King and Rebelo (1990), \(\tau^{1*} = 0.197\). This measure does not take account of the growth rate of the government expenditure, and therefore is independent of the base year growth rate (thus independent of the base year tax rate).

Define \(G \equiv \theta = \text{constant}\). In the interpretation (d), equation (8) can be modified as

\[
\int_0^\infty \theta Y e^{-\gamma_1 t} dt = \int_0^\infty \tau^1 Y e^{-\gamma_1 t} dt.
\]

Thus, \(\theta = \tau^1\). We can always increase the government expenditure-income ratio by raising \(\tau^1\). Even if the economy’s growth rate is negative as a result of raising the tax rate, the government expenditure-income ratio always rises. Thus, this measure is not useful.

In the following sections, we inquire above questions in the general two sector model of endogenous growth focusing mainly on the interpretation (b) and (c), taking account of above conditions of Stokey and Rebelo (1995). We analyze not only the steady state, but also the transition path in order to appropriately capture the interest rate effect in calculating the present value, as discussed above. The Laffer curves are derived by numerical simulations in the general model under plausible parameter values.

### III. Model

This section describes the model.\(^{40}\) It is a discrete time version of a standard two-sector model of endogenous growth with human and physical capital. It closely follows Pecorino (1995).

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\(^{36}\)Parameter \(A\) must be recalculated for consistency with the assumption. In this case, \(A\) is determined such that \((A - \delta)(1 - \tau^0) = \gamma^0\). That is, \(A = \frac{\gamma^0}{\tau^0} + \delta = \frac{0.032}{0.15} + 0.1 = 0.14\).

\(^{37}\)In the more general two sector model, different tax treatments of depreciation allowances across sectors may be assumed. For example, Pecorino (1995) and this paper assume full tax deductibility of physical capital depreciation expense and no tax deductibility of human capital depreciation expense. Since \(K\) is a composite of physical and human capital, the true \(\tau^{1*}\) lies in between 0.136 and 0.25. The result depends on the relative share of physical and human capital.

\(^{38}\)Assume that \(\gamma_1 - \nu^0 > 0\).

\(^{39}\)Equation (18) can be also expressed as \(G = \frac{AK_0}{\rho} \tau^1 = \frac{AK_0}{1 - \tau^1}\). Thus, Pecorino (1995)'s measure (4) may be roughly interpreted as trying to partially capture the concept (c).

\(^{40}\)From this section, we employ a discrete-time model mainly for computational purposes.
A. Firm

There are many competitive firms. The economy consists of 2 sectors: sector 1 produces the consumption and physical capital goods, and sector 2 produces human capital. Technology is described by the following constant returns to scale CES production function:

\( Y^1 = F^1(\phi^1_jK^1_j, u^1_jH^1_j) = A^1[\alpha^1_K(\phi^1_jK^1_j)^{\psi_1} + (1 - \alpha^1_K)(u^1_jH^1_j)^{\psi_1}]^{\frac{1}{\psi_1}}, \)

\( Y^2 = F^2(\phi^2_jK^2_j, u^2_jH^2_j) = A^2[\alpha^2_K(\phi^2_jK^2_j)^{\psi_2} + (1 - \alpha^2_K)(u^2_jH^2_j)^{\psi_2}]^{\frac{1}{\psi_2}}, \)

where \( K \) is the total physical capital stock, \( H \) the total human capital stock, \( Y^j \) the output in sector \( j \), \( \phi^j \) the ratio of the physical capital stock devoted to the production in sector \( j \), \( u^j \) the portion of time devoted to the production in sector \( j \), \( A^j \) the scale parameter, \( \alpha^j_K \) the physical capital share parameter, and \( \psi_j \) the physical-human capital substitution parameter of sector \( j \)'s production function, respectively \( (j = 1, 2) \).

Let \( l \) be the leisure time. Normalize the time endowment to 1. Then we have the following constraints on capital utilization and time:

\( \phi^1_t + \phi^2_t = 1, \)

\( u^1_t + u^2_t + l_t = 1. \)

Assume for simplicity full tax deductibility of physical capital depreciation expense and no tax deductibility of human capital depreciation expense.\(^{41}\) The after-tax rate of return to each type of capital must be equalized across sectors:

\( (1 - t^1_K)(r^1_t - \delta_K) = (1 - t^2_K)(r^2_t - \delta_K), \)

\( (1 - t^1_H)w^1_t = (1 - t^2_H)w^2_t. \)

where \( t^j_k \) is the flat tax rate\(^{42}\) on the rate of return in sector \( j \) of \( h \)-type capital, \( \delta_h \) the depreciation rate of \( h \)-type capital, \( r^j \) the rate of return to physical capital in sector \( j \), and \( w^j \) the rate of return to human capital in sector \( j \) \( (j = 1, 2 \text{ and } h = K, H) \).

Firms maximize their own profits. First-order conditions for firms are as follows:\(^{43}\)

\( p^1_t F^1_t = p^1_t r^1_t, \)

\( p^2_t F^2_t = p^2_t r^2_t, \)

\( p^1_t F^2_t = p^2_t w^1_t, \)

\( p^1_t F^2_t = p^2_t w^2_t, \)

where \( p^j \) is the price of output in sector \( j \), and \( F^j_n \) is the derivative of \( F^j \) with respect to the \( n \)-th variable. Normalize \( p^1 \) to 1. Then equations (27) to (30) can be simplified as follows:\(^{44}\)

\( F^1_t = r^1_t, \)

\( F^2_t = r^2_t, \)

\( F^1_t = p^1_t w^1_t, \)

\( F^2_t = p^2_t w^2_t. \)

\(^{41}\)This assumption is the same as that of Pecorino (1995).

\(^{42}\)This paper only analyzes the flat-rate tax. This feature is very common in many related literature, such as Lucas (1990), Ireland (1994), Pecorino (1995), Stokey and Rebelo (1995), and so on.

\(^{43}\)We can consider that firms solve the following static maximization problem because there is no adjustment cost.

\[
\max_{K^1, H^1} \Pi^1 = p^1 F^1(K^1, H^1) - p^1 r^1 K^1 - p^2 w^1 H^1,
\]

\[
\max_{K^2, H^2} \Pi^2 = p^2 F^2(K^2, H^2) - p^1 r^2 K^2 - p^2 w^2 H^2.
\]

\(^{44}\)Then, \( p^2 \) can be interpreted as the relative value of the human capital in terms of the physical capital (or the consumption goods).
B. Consumer

An infinitely-lived representative household with perfect foresight solves the following problem, taking the paths of future prices and lump-sum transfers from the government as given:

\[
(31) \quad \max_{c_t, l_t} V = \sum_{t=0}^{\infty} \left( \frac{1}{1 + \rho} \right)^t U(c_t, l_t)
\]

subject to

\[
\begin{align*}
(32) \quad K_{t+1} - K_t &= (1 - t^H_1)(r^1_t - \delta_K)(\phi^1_t K_t) + (1 - t^H_2)(r^2_t - \delta_K)(\phi^2_t K_t) \\
&\quad + (1 - t^H_1)p^2_t w^1_t (u^1_t H_t) + (1 - t^H_2)p^2_t w^2_t (u^2_t H_t) \\
&\quad + Z_t - (1 + t_c)c_t - p^2_t I^H_t \\
H_t &= (1 - t^H_1)(r^1_t - \delta_K)K_t + (1 - t^H_2)p^2_t w^1_t (1 - l_t)H_t \\
&\quad + Z_t - (1 + t_c)c_t - p^2_t I^H_t,
\end{align*}
\]

where \( V \) is the intertemporal utility function, \( U \) the instantaneous utility function, \( \rho \) the rate of time preference, \( c \) the consumption, \( Z \) the lump sum transfer from the government, \( l_t \) the consumption tax rate, and \( I^H \) the gross investment in human capital. The second equality in equation (32) follows from (23), (24), (25), and (26). It is assumed that the rate of population growth is 0; thus, all variables can be interpreted as levels per capita.

First-order conditions of this problem are as follows:\(^{45}\)

\[
\begin{align*}
(34) \quad \frac{U_c(c_{t+1}, l_{t+1})}{U_c(c_t, l_t)} &= \frac{1 + \rho}{1 + (1 - t^H_1)(r^1_{t+1} - \delta_K)} , \\
(35) \quad \frac{U_l(c_{t+1}, l_{t+1})}{U_c(c_t, l_t)} &= \frac{(1 - t^H_1)p^2_t w^1_t H_t}{1 + t_c} , \\
(36) \quad 1 + (1 - t^H_1)(r^1_{t+1} - \delta_K) &= [1 + (1 - t^H_1)w^1_{t+1}(1 - l_{t+1}) - \delta_H] \frac{p^2_{t+1}}{p^2_t} ,
\end{align*}
\]

where \( U_c \) and \( U_l \) are the marginal utility of consumption and leisure, respectively. Equation (34) is a standard Euler equation. Equation (35) states that the ratio of marginal utility of leisure to that of consumption is equal to the after-tax real wage in efficiency units. Equation (36) states that the returns from investing in physical capital versus human capital should be the same.\(^{46}\)

Specify the utility function to be of the following form:\(^{47}\)

\[
(37) \quad U(c_t, l_t) = \frac{1}{1 - \sigma}(c_t^{1-\sigma})^{1-\sigma}.
\]

Then, (34) and (35) become

\[
\begin{align*}
(34') \quad \left( \frac{c_{t+1}}{c_t} \right)^\sigma \left( \frac{l_{t+1}}{l_t} \right)^{(\sigma - 1)} &= \frac{1 + (1 - t^H_1)(r^1_{t+1} - \delta_K)}{1 + \rho} , \\
(35') \quad \frac{\Omega_c}{l_t} &= \frac{(1 - t^H_1)p^2_t w^1_t H_t}{1 + t_c} .
\end{align*}
\]

\(^{45}\)Transversality conditions are

\[
\begin{align*}
\lim_{t \to \infty} \Pi_{s=1}^{t} \left[ 1 + (1 - t^H_2)(r^2_t - \delta_K) \right] &= 0 , \quad \text{and} \\
\lim_{t \to \infty} \Pi_{s=1}^{t} \left[ 1 + (1 - t^H_1)(r^1_t - \delta_K) \right] &= 0 .
\end{align*}
\]

\(^{46}\)Suppose that you are thinking about investing $1 in physical capital versus human capital. If you invest $1 in physical capital in this period, you can get 1 + (1 - t^H_1)(r^1_{t+1} - \delta_K) dollar in the next period. If you invest 1 unit in human capital in this period, you will get \( p^2_{t+1} \left[ 1 + (1 - t^H_2)w^1_{t+1}(1 - l_{t+1}) - \delta_H \right] \) in the next period. However, since the price of human capital in this period is \( p^2_t \), you can invest 1/p^2_t by $1. Therefore by investing $1 in human capital in this period, you can get \( p^2_{t+1} \left[ 1 + (1 - t^H_2)w^1_{t+1}(1 - l_{t+1}) - \delta_H \right] \) in the next period. Returns from both activities should be the same at the optimum. This gives equation (36).

\(^{47}\)Under this specification, the utility boundedness is guaranteed by the condition \( \frac{(1+\nu)^{1-\sigma}}{1+\rho} < 1 \), where \( \nu \) is the economy’s growth rate. This corresponds to the condition M in Bond et al. (1996).
C. Government

Government tax revenue in each period $TR_t$ is

\[ TR_t = t^K(r^K_1 - \delta_K)(\phi^1_K + t^K_H(p^2_K u^1_H + t^K_H u^2_H) + t^K_H u^2_H) + t_c, \]

Government is assumed to return the constant ratio $1 - \eta$ of its revenue in a lump sum manner,\(^{48}\)

\[ Z_t = (1 - \eta) TR_t. \]

The rest of the tax revenue, $\eta TR_t$, is assumed to be equal to the government consumption, $G_t$, that does not affect the marginal rates of substitution among private goods. The government budget is balanced for each period.

D. Material Balance Condition

Equations (32), (38), and (39) together imply goods market clearing for sector 1,

\[ Y_t^1 = c_t + K_{t+1} = K_t + \delta_K K_t + \eta TR_t. \]

In sector 2, output is equal to the gross investment in human capital,

\[ I_t^H = Y_t^2. \]

E. Equilibrium

Equations (21), (22), (23), (24), (25), (26), (27'), (28'), (29'), (30'), (32), (33), (34'), (35'), (36), (38), (39), and (40) constitute a system of the nonlinear simultaneous difference equations for the whole economy. The number of equations is 18, while endogenous variables are

\[ K, H, c, p^2, Y^1, Y^2, \phi^1, \phi^2, u^1, u^2, l, r^1, r^2, w^1, w^2, I^H, Z, \text{and } TR. \]

This economy basically consists of four difference equations (32), (33), (34'), and (36), and basic variables $c, H, K,$ and $p^2$. Other equations give all other variables as functions of the basic variables. It is much easier to analyze the transitional dynamics of this system by reducing the number of the difference equations, as shown below.

Define $a_t \equiv K_t$, and $b_t \equiv \frac{c_t}{y_t}$. Then we will show that we can describe the economy by 3 difference equations (in 3 variables $a, b,$ and $p^2$). Define $y_t^1 \equiv \frac{Y_t^1}{y_t}, y_t^2 \equiv \frac{Y_t^2}{y_t}, m_t \equiv \frac{I_t^H}{y_t}$, and $n \equiv \frac{TR}{y_t}$. We can transform our system of equations into

\[
\begin{align*}
  y_t^1 &= f^1\left(\frac{\phi^1_t}{u^1_t} a_t, 1\right) = f^1\left(\frac{\phi^1_t}{u^1_t} a_t\right) = A^1[\alpha^K_t\left(\frac{\phi^1_t}{u^1_t} a_t\right)^{\psi_1} + (1 - \alpha^K_t)]^{\frac{1}{\psi_1}}, \\
  y_t^2 &= f^2\left(\frac{\phi^2_t}{u^2_t} a_t, 1\right) = f^2\left(\frac{\phi^2_t}{u^2_t} a_t\right) = A^2[\alpha^K_t\left(\frac{\phi^2_t}{u^2_t} a_t\right)^{\psi_2} + (1 - \alpha^K_t)]^{\frac{1}{\psi_2}}, \\
  \phi_t^1 + \phi_t^2 &= 1, \\
  a_t^1 + a_t^2 + l_t &= 1, \\
  (1 - t^K_K)(r_t^1 - \delta_K) &= (1 - t^K_H)(r_t^2 - \delta_K), \\
  (1 - t^K_H)u_t^1 &= (1 - t^K_H)u_t^2, \\
  r_t^1 &= f^{1f}\left(\frac{\phi^1_t}{u^1_t} a_t\right), \\
  r_t^2 &= p_t^2 f^{2f}\left(\frac{\phi^2_t}{u^2_t} a_t\right),
\end{align*}
\]

\(^{48}\)Consider the following 2 cases. (1) $\eta = \text{const.} > 0$ such that the actual level of $G$ in the base year is equal to $\eta TR$. Since the tax rate that affects the margin is the marginal tax rate, $t^K_1$ can be interpreted as the marginal tax rate. However, then, we cannot obtain the tax revenue by multiplying the marginal tax rate by the tax base. However, we can approximate the actual level of tax revenue by the marginal tax rate times the tax base minus the lump sum transfer. (2) $\eta = 0$. In this case, all tax revenues are returned as a lump sum transfer. This is the same experiment as in Stokey and Rebelo (1995). This experiment is often analyzed to focus on the effects of taxation. This case is also analyzed later.
boundedness is guaranteed by the condition
\[ (55) \]
\[ t_{K1}^1 (r_{1}^1 - \delta_{K}) = (1 + t_{K2}^2 - \delta_{H}) p_t^1 w_t^1 (1 - l) \]
\[ + (1 - \eta) n_t - (1 + t_c) b_t - p_t^2 m_t, \]
\[ (56) \]
\[ (1 + t_{K1}^1) m_{t+1} = 1 (1 + t_{K2}^2 - \delta_{H}) p_t^2, \]
Thus the number of equations is 16, and the endogenous variables are
\[ a, b, p^2, y^1, y^2, \phi^1, \phi^2, u^1, u^2, t, r^1, r^2, w^1, w^2, m, \] and \( n \).

**F. Steady State Equilibrium**

Set the values of the period \( t + 1 \) variables equal to those of the period \( t \) variables in equations (41) to (56).

Then, the steady state equilibrium of this economy is characterized by the following 16 equations:

\[ (41') \]
\[ y^1 = f^1 \left( \frac{\phi^1}{u^1} a, 1 \right) = f^1 \left( \frac{\phi^1}{u^1} a \right) = A^1 \left[ \alpha^1_K \left( \frac{\phi^1}{u^1} a \right) \psi + (1 - \alpha^1_K) \right] \frac{a^2}{a^1}, \]
\[ (42') \]
\[ y^2 = f^2 \left( \frac{\phi^2}{u^2} a, 1 \right) = f^2 \left( \frac{\phi^2}{u^2} a \right) = A^2 \left[ \alpha^2_K \left( \frac{\phi^2}{u^2} a \right) \psi + (1 - \alpha^2_K) \right] \frac{a^2}{a^2}, \]
\[ (43') \]
\[ \phi^1 + \phi^2 = 1, \]
\[ (44') \]
\[ u^1 + u^2 + l = 1, \]
\[ (45') \]
\[ (1 + t_{K1}^1)(r_{1}^1 - \delta_{K}) = (1 + t_{K2}^2)(r_{2}^1 - \delta_{K}), \]
\[ (46') \]
\[ (1 - t_{K1}^1) w^1 = (1 - t_{K2}^2) w^2, \]
\[ (47') \]
\[ r^1 = f^{1*} \left( \frac{\phi^1}{u^1} a \right), \]
\[ (48') \]
\[ r^2 = p^2 f^{2*} \left( \frac{\phi^2}{u^2} a \right), \]
\[ (49') \]
\[ p^2 w^1 = f^1 \left( \frac{\phi^1}{u^1} a \right) - \left( \frac{\phi^1}{u^1} a \right) f^{1*} \left( \frac{\phi^1}{u^1} a \right), \]
\[ (50') \]
\[ w^2 = f^2 \left( \frac{\phi^2}{u^2} a \right) - \left( \frac{\phi^2}{u^2} a \right) f^{2*} \left( \frac{\phi^2}{u^2} a \right), \]
\[ (51') \]
\[ ((1 - t_{K1}^1)(r_{1}^1 - \delta_{K}) + \delta_{K} - m) a + (1 - t_{K2}^2) p^2 w^1 (1 - l) + (1 - \eta) n - (1 + t_c) b - p^2 m = 0, \]
\[ (52') \]
\[ (1 - t_{K1}^1)(r_{1}^1 - \delta_{K}) = (1 - t_{K2}^2) w^1 (1 - l) - \delta_{H}, \]
\[ (53') \]
\[ (1 + \rho) (m + (1 - \delta_{H}) p_t^2) = 1 + (1 - t_{K1}^1)(r_{1}^1 - \delta_{K}), \]
\[ (54') \]
\[ \frac{\Omega b}{l_t} = \frac{(1 - t_{K2}^2) p_t^2 w^1}{1 + t_c}, \]
\[ (55') \]
\[ t_{K1}^1 (r_{1}^1 - \delta_{K}) \phi^1 a + t_{K2}^2 (r_{2}^1 - \delta_{K}) \phi^2 a + t_{K1}^1 p_t^2 w^1 u_t^2 + t_{K2}^2 p_t^2 w^2 u_t^2 + t_c b = n, \]
\[ (56') \]
\[ m = y^2 u^2, \]

where each variable without time subscript denotes the steady state value.\(^{49}\)

\(^{49}\)Note that (53') is transformed as
\[ \frac{1 + (m - \delta_{H})}{1 + (l - t_{K1}^1)(r_{1}^1 - \delta_{K})} = \frac{1 + (m - \delta_{H})}{1 + (1 - t_{K1}^1)(r_{1}^1 - \delta_{K})}. \]

Since the growth rate is given by \( m - \delta_{H} \), the boundedness is guaranteed by the condition
\[ \frac{1 + (m - \delta_{H})}{1 + (1 - t_{K1}^1)(r_{1}^1 - \delta_{K})} < 1. \]

See footnote (47) and TVC’s in footnote (45).
G. Dynamics of the economy

Consider equations (41), (42), (43), (44), (45), (46), (47), (48), (49), (50), (54), (55), and (56). Clearly, given variables $a_t$, $b_t$, and $p_t^2$, we can obtain 13 variables $y^1, y^2, \phi^1, \phi^2, u^1, w^1, l, r^1, r^2, w^1, w^2, m,$ and $n$ by solving these 13 equations. Therefore, these 13 variables can be expressed as functions of $a_t, b_t,$ and $p_t^2,$ for example,

$$y^1_t = y^1_t(a_t, b_t, p_t^2), \ldots$$

Next, consider equations (51), (52), and (53). By substituting above functions into these 3 equations, they can be transformed as follows:

$$m_t(a_t, b_t, p_t^2) + (1 - \delta_H)a_{t+1} = (1 + (1 + t_H^1)(r^1_t(a_t, b_t, p_t^2) - \delta_K))a_t + (1 - t_H^1)p_t^2 w^1_t(a_t, b_t, p_t^2)(1 - l_t(a_t, b_t, p_t^2))$$
$$+ (1 - \eta)m_t(a_t, b_t, p_t^2) - (1 + t_e) b_t - p_t^2 m_t(a_t, b_t, p_t^2),$$

$$[1 + (1 - t_H^1)w^1_{t+1}(a_{t+1}, b_{t+1}, p_{t+1}^2)(1 - l_{t+1}(a_{t+1}, b_{t+1}, p_{t+1}^2) - \delta_H)] p_{t+1}^2$$
$$= [1 + (1 - t_K^1)(r^1_{t+1}(a_{t+1}, b_{t+1}, p_{t+1}^2) - \delta_K)] p_t^2,$$

$$\left(\frac{b_{t+1}}{b_t}\right)^{-\sigma} \left(\frac{l_{t+1}(a_{t+1}, b_{t+1}, p_{t+1}^2)}{t_{t+1}(a_t, b_t, p_t^2)}\right)^{\sigma} \Omega(1 - \sigma) = \frac{(1 + \rho)(m_t(a_t, b_t, p_t^2) + (1 - \delta_H))^\sigma}{1 + (1 - t_K^1)(r^1_{t+1}(a_{t+1}, b_{t+1}, p_{t+1}^2) - \delta_K)}.$$

Therefore, the system of (57), (58), and (59) can describe the motion of this economy.

Define $X_t \equiv \begin{pmatrix} a_t \\ b_t \\ p_t^2 \end{pmatrix}.$ Then, this system can be simply expressed as

$$\Gamma(X_{t+1}, X_t) = 0,$$

where $\Gamma(X_{t+1}, X_t) \equiv \begin{pmatrix} \Gamma^1(X_{t+1}, X_t) \\ \Gamma^2(X_{t+1}, X_t) \\ \Gamma^3(X_{t+1}, X_t) \end{pmatrix}$, and a function $\Gamma^1$ corresponds to (57), $\Gamma^2$ to (58), and $\Gamma^3$ to (59).

That is,

$$\Gamma^1 \equiv \left( m_t(a_t, b_t, p_t^2) + (1 - \delta_H) \right) a_{t+1} - (1 + (1 - l_K^1)(r^1_t(a_t, b_t, p_t^2) - \delta_K)) a_t$$
$$- (1 - t_H^1)p_t^2 w^1_t(a_t, b_t, p_t^2)(1 - l_t(a_t, b_t, p_t^2))$$
$$- (1 - \eta)m_t(a_t, b_t, p_t^2) + (1 + t_e) b_t + p_t^2 m_t(a_t, b_t, p_t^2) = 0,$$

$$\Gamma^2 \equiv [1 + (1 - t_H^1)w^1_{t+1}(a_{t+1}, b_{t+1}, p_{t+1}^2)(1 - l_{t+1}(a_{t+1}, b_{t+1}, p_{t+1}^2) - \delta_H)] p_{t+1}^2$$
$$- [1 + (1 - t_K^1)(r^1_{t+1}(a_{t+1}, b_{t+1}, p_{t+1}^2) - \delta_K)] p_t^2 = 0,$$

$$\Gamma^3 \equiv \left( \frac{b_{t+1}}{b_t}\right)^{-\sigma} \left(\frac{l_{t+1}(a_{t+1}, b_{t+1}, p_{t+1}^2)}{t_{t+1}(a_t, b_t, p_t^2)}\right)^{\sigma} \Omega(1 - \sigma) \left[1 + (1 - t_K^1)(r^1_{t+1}(a_{t+1}, b_{t+1}, p_{t+1}^2) - \delta_K)\right]$$
$$- (1 + \rho)(m_t(a_t, b_t, p_t^2) + (1 - \delta_H))^\sigma = 0.$$

Take a first-order Taylor-series expansion of the equation (60) at the steady state, and obtain

$$\Gamma(X^*, X^*) + \frac{\partial \Gamma(X^*, X^*)}{\partial X_{t+1}}(X_{t+1} - X^*) + \frac{\partial \Gamma(X^*, X^*)}{\partial X_t}(X_t - X^*)$$
$$= \frac{\partial \Gamma^1(X^*, X^*)}{\partial X_{t+1}}(X_{t+1} - X^*) + \frac{\partial \Gamma(X^*, X^*)}{\partial X_t}(X_t - X^*) = 0.$$

Therefore this economy can be approximately characterized by the following first-order linear difference equation system in the $\varepsilon$-neighborhood of the steady state.

$$X_{t+1} - X^* = M(X_t - X^*),$$

50The linearization in this section follows Laitner (1995).
where \( M = -\left[ \frac{\partial(x^{*},x^{*})}{\partial x_{i+1}} \right]^{-1} \frac{\partial x}{\partial x_i} \). This matrix \( M \) can give the information about the local stability of this economy.\(^{51}\)

Since \( a_0 = \frac{\partial x}{\partial t} \) is a historical variable and both \( b_i = \frac{\partial x}{\partial t} \) and \( p^2 \) are jumping variables, there are three cases depending on the modulus of eigenvalues of the matrix \( M \).

Case 1. Two or more eigenvalues of \( M \) have modulus less than one
In this case, indeterminacy occurs.

Case 2. One has modulus less than one and two have modulus greater than one
In this case, there is one and only one equilibrium path converging to the steady state for any \( a_0 \), i.e., \( H_0 \) and \( K_0 \).

Case 3. All have modulus greater than one
In this case, the system is unstable.

Table 1 reports eigenvalues in the neighborhoods of new steady states corresponding to different tax rates. Case 2 always applies under plausible parameters. The general solution of this system of first order linear difference equations is \( X_t - X^* = c_1\lambda_1v_1 + c_2\lambda_2v_2 + c_3\lambda_3v_3 \), where \( c_1 \), \( c_2 \), and \( c_3 \) are constants, \( \lambda_1 \), \( \lambda_2 \), and \( \lambda_3 \) eigenvalues, and \( v_1 \), \( v_2 \), and \( v_3 \) eigenvectors. Since the system is saddle path stable, we can assume that only \( \lambda_1 \) has modulus less than one and that \( c_3 = 0 \) along the saddle path. Then the motion of the economy along the saddle path is described by equation \( X_t - X^* = \lambda_1(X_0 - X^*) \). This equation suggests that \( \lambda_1 \) has an inverse relationship with the convergence speed of the economy. That is, the larger \( \lambda_1 \), the slower the economy converges to the steady state. Table 1 shows that as the tax rate becomes higher, \( \lambda_1 \) becomes larger and approaches toward 1. This means that as the tax rate becomes higher, the economy converges more slowly to the steady state. Table 1 also shows that the period during which 99.9% adjustment of a historical variable \( a \) is made is longer as the tax rate becomes higher. It also shows that as the tax rate becomes higher, \( \lambda_3 \) becomes smaller and approaches toward 1 from above, and \( \lambda_2 \) roughly remains constant near (but greater than) 1. Thus the economy might move from the relatively stable phase to the relatively unstable one as the tax rate becomes high. In a similar model without labor supply decisions, Bond et al. (1996) show that instability or indeterminacy emerges when factor taxes are too distortionary. The results in Table 1 seem consistent with their results.\(^{52}\)

The two sector endogenous growth model is known to exhibit complex dynamics. As shown in figure 1O, 1P, and 5P below, physical capital intensity in sector 1 is always greater than that in sector 2 under the parameter values assumed in this paper. That is, the educational sector is always human capital intensive. Thus, the economy is in the case where the price adjustment process is stable. The relative price of human capital, \( p^2 \), adjusts during a transition to offset the unstable adjustment process of output at fixed prices.\(^{53}\)

### IV. Simulation Results

It would be desirable to solve the difference equations analytically. Then the derived analytical expressions can give us economic interpretations. Unfortunately, however, it is well known that this system cannot be solved analytically because of the nonlinearity of the system. Thus the nonlinear system must be solved numerically.\(^{54}\)

The same tax structure as that of Pecorino (1995) is assumed for comparisons: In sector 1, the wage tax rate and the physical capital income tax rate are assumed to be the same as those of Lucas (1990), i.e., 40% and 36%, respectively, while in the educational sector (sector 2) the wage tax rate is 1/3 of sector 1’s to take account of “the extent to which the human capital input into human capital accumulation reflects foregone earnings (which are untaxed)”, and the physical capital income is free from taxation “to proxy for the nonprofit nature of higher education”.\(^{55}\) In addition, consumption tax rate is assumed to be 5% for the government budget balance. Tax rates are assumed to be changed, keeping this proportion. That is,

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51 How to compute the matrix \( M \) is explained in Appendix D in detail.

52 However, their tax structure is different from ours. In their model, physical capital income tax rate in sector 1 and 2 are assumed to be the same, and no consumption tax is assumed. This may preclude the emergence of instability or indeterminacy here within the range of the tax rate change considered in this paper.

53 When the educational sector is physical capital intensive, the price adjustment process is unstable. The relative price \( p^2 \) must jump to its SSE value immediately, and output adjustment process at fixed prices becomes stable. See Bond et al. (1996) in more detail.

54 The computational procedure of this numerical analysis is explained in Appendix E in detail.

\( t_K^1 = (0.9) t_H^1, \ t_H^2 = (1/3) t_H^1, \ t_K^2 = 0 \cdot t_H^1, \) and \( t_c = (0.125) t_H^1. \) Hereafter, the word “tax rate” means \( t_H^1. \) When the tax rate \( t_H^1 \) changes, all other tax rates also change in this proportion at the same time.

For simplicity, assume that at first the economy stays in the steady state under the tax parameters described above.\(^56\) Suppose that at time 0 there is an unexpected, abrupt, and permanent change in the tax rate. Given saddle point stability, the economy jumps to the point along the transition path with historical variables fixed, and then moves gradually toward the new steady state.\(^57\)

Figure 1A to 1P show relationships between tax rates and the corresponding steady-state variables. As the tax rate becomes high, leisure time monotonically increases,\(^58\) and both after-tax interest rate and after-tax wage rate monotonically decreases.\(^59\) Therefore, from (53'), the long-run growth rate\(^60\) monotonically decreases. Figure 2A and table 2 show the relationship between tax rates and the long-run growth rate.\(^61\) A 5% decrease in the average marginal tax rate\(^62\) from 40% increases the growth rate by 0.21%. This is within the range of the empirical analysis of Engen and Skinner (1996) who find that “the evidence is consistent with lower taxes having modest positive effects on economic growth”.\(^63\) The long-run growth rate is determined by the human capital investment, and therefore the production in sector 2 must decrease monotonically. It means that both \( \phi^2 \) and \( u^2 \) monotonically decrease and approach 0, and as a result, \( \phi^1 \) approaches 1, and \( u^1 \) monotonically decreases. Thus, the physical capital moves from sector 2 to sector 1, and the portion of human capital devoted to production in sector 1 and 2 decreases. As a result, the tax base shrinks.\(^64\)

Note that physical capital intensity in sector 1 is always greater than that in sector 2. Thus, this economy is always in the case where the educational sector is human capital intensive. Figure 2B shows the relationships among tax rates, before-tax interest rate net of depreciation in sector 1, and after-tax interest rate net of depreciation across steady states.\(^65\) After-tax interest rate net of depreciation monotonically decreases as the tax rate becomes high, and both net-of-depreciation interest rates become negative when the tax rate is beyond 90%. Figure 2C shows the relationships among tax rates, the tax revenue-GNP ratio, and the tax revenue-potential GNP ratio across steady states.\(^66\) Both ratios monotonically increase at first, but turn down when the tax rate is beyond 90%, since the tax revenue from physical capital income becomes negative by the assumption of full tax deductibility of physical capital depreciation expense.

Next, the government’s present value budget constraint is analyzed, taking account of the transition path toward the steady state after the change in the tax rate. First, tax revenues in different tax regimes are compared in terms of the interpretation (a).\(^67\) The present value of tax revenue collections discounted by the after-tax interest rate net of depreciation is expressed as follows:\(^68\)

\[
PVR = \sum_{t=0}^{\infty} \frac{TR_t}{\prod_{s=1}^{t} (1 + \gamma_s)},
\]

where \( \gamma_t = (1 - t_K^1(K_t^1 - \delta_K)) = (1 - t_K^2(K_t^2 - \delta_K)). \)

Figure 3 shows the relationship between the tax rate and the equation (63).\(^69\) The present value of tax revenues monotonically increases until the tax rate is raised to 90%. Even if the growth rate is negative beyond 70% tax rate\(^70\) as figure 2A shows, the present value continues to increase until the tax rate is raised.

\(^{56}\)Other parameter values are also assumed to be the same as those of Pecorino (1995). See Appendix C for other parameter values.

\(^{57}\)As an example, time paths of several main variables during a transition when the tax rate is raised to 46% are shown in figure 5A to 5P below.

\(^{58}\)The substitution effect dominates the income effect since about half of tax revenues are rebated in a lump-sum fashion (in the later analysis, all tax revenues are returned as a lump-sum transfer).

\(^{59}\)This is clear from figure 1J and figure 1L.

\(^{60}\)The long run growth rate is given by \( m^* - \delta_H. \)

\(^{61}\)The relationship between tax policies and the long-run growth rate is analytically explained in detail in Rebelo (1991) and Stokey and Rebelo (1995). Growth rate effects of other aspects of tax reform (for example, replacing the income tax with a consumption tax) in a similar model is analyzed in Pecorino (1994).

\(^{62}\)Initial labor income tax rate in Lucas (1999) can be considered as the average marginal tax rate. “(T)hink of all labor income as being taxed at a higher rate and then to treat the difference between labor income tax revenues at this higher rate and actual revenues and a lump-sum rebated back to consumers” (Lucas (1996), p.307).


\(^{64}\)See Appendix B for this effect.

\(^{65}\)Since physical capital income in sector 2 is assumed to be free from taxation, after-tax interest rate net of depreciation is equal to before-tax interest rate net of depreciation in sector 2.

\(^{66}\)GNP is defined as \( Y = p^2 Y^2. \)

\(^{67}\)See section II about the interpretation (a) to (d).

\(^{68}\)Brock and Turnovsky (1981) show that the government’s present value budget constraint discounted by the after-tax interest rate matters.

\(^{69}\)The present value of tax revenues in the case where there is no change in the tax rate and the economy continues to stay at the original steady state, is normalized to 1.

\(^{70}\)That is, the condition for perpetual growth does not hold when the tax rate is beyond 70%. For perpetual growth, the
to 90%. Unlike the simple model of section II, there exists a turning point\textsuperscript{71} in this general model. It is only because the before-tax interest rate net of depreciation becomes negative beyond 90% tax rate as figure 2B shows, and therefore the tax revenue from physical capital income is negative, i.e., physical capital is actually subsidized since full tax deductibility of physical capital depreciation expense is assumed. This channel does not exist in the model of section II since $A - \delta$ is always positive and constant. Except this point, the basic mechanism is the same as in the model of section II. Thus, checking this turning point is not meaningful. As to the interpretation (d), in figure 2C, in the steady state the tax revenue-GNP ratio monotonically increases at first, but turns down when the tax rate is beyond 90% because of the same reason as above. We can always increase the tax revenue-GNP ratio as far as the tax base is positive. The basic mechanism is also the same as in the model of section II. The above argument clearly shows that comparing tax revenues in terms of the interpretation (a) or (d) is not meaningful in a framework of endogenous growth.

By assuming that all tax revenues are returned in a lump sum manner, i.e., by setting $\eta = 0$, the same experiment as that of Ireland (1994)\textsuperscript{72} can be analyzed.\textsuperscript{73} Assuming that the initial government debt is 0, the government’s present value budget constraint in terms of the interpretation (b) becomes\textsuperscript{74}

\begin{equation}
\sum_{t=0}^{\infty} \frac{(g + \Delta g)H_0[(1 - \delta_H) + \eta]}{\Pi_{s=1}^{t}(1 + \gamma_s)} = \sum_{t=0}^{\infty} \frac{\hat{n}_t \hat{H}_t}{\Pi_{s=1}^{t}(1 + \gamma_s)},
\end{equation}

where $-$ denotes the old steady state variables, and $\wedge$ does the variables moving along the transition path toward the steady state after the change in the tax rate. Figure 4A and table 3A show the relationship between tax rates and the rate of change in the constant government expenditure per efficiency units, $\Delta g/g$. This is maximized at 46%, and then we can increase $g$ by 2.08%. Therefore, in this general case, a tax-cut is not self-financing in terms of (b). In terms of the interpretation (c), the government’s present value budget constraint becomes\textsuperscript{75}

\begin{equation}
\sum_{t=0}^{\infty} \frac{G}{\Pi_{s=1}^{t}(1 + \gamma_s)} = \sum_{t=0}^{\infty} \frac{\hat{n}_t \hat{H}_t}{\Pi_{s=1}^{t}(1 + \gamma_s)}.
\end{equation}

Figure 4B and table 3B show the relationship between tax rates and the rate of change in the constant government expenditure $G$ that can be financed, $\Delta G/\bar{G}$, where $\bar{G}$ is the original steady state value. This is maximized at 53%, and then we can increase $G$ by 9.55%.

Raising the tax rate increases the distortion, and therefore generates the welfare loss. Pareto optimality is achieved when the tax rate is 0%.\textsuperscript{76} As the tax rate is raised, not only intertemporal decisions, intratemporal consumption-leisure decisions, and physical-human capital investment decisions are distorted, but also intersectoral misallocations of both capital are distorted because of the tax structure assumed in this paper. The intersectoral misallocations of both capital are worse and worse as the tax rate becomes high, and therefore, the welfare loss generated increases more than proportionately.\textsuperscript{77} The welfare loss is measured in terms of the measure of Lucas (1990) and King and Rebelo (1990). That is, it is measured in terms of a “compensating consumption supplement” $\zeta$.\textsuperscript{78}

\begin{equation}
V\left(\{1 + \zeta \hat{c}_t, \hat{l}_t\}_{t=0}^{\infty}\right) = V\left(\{\hat{c}_t, \hat{l}_t\}_{t=0}^{\infty}\right),
\end{equation}

where $V$ is the intertemporal utility, $\{\hat{c}_t, \hat{l}_t\}_{t=0}^{\infty}$ the time path of consumption and leisure at the old steady state, and $\{\hat{c}_t, \hat{l}_t\}_{t=0}^{\infty}$ the time path of consumption and leisure when the tax rate is changed. By compensating consumption at the old steady state by $\zeta \times 100(\%)$, the representative agent can achieve the same utility as that when the tax rate is changed. Figure 4C and table 3C show the relationship between tax rates and
When the tax rate is raised to 46%, the welfare loss is equivalent to 5.5% decrease in consumption every period. If the tax rate is raised to 53% to maximize $g/G$, then the welfare loss is equivalent to about 13% decrease in consumption every period.

Overall, unlike the result of Ireland (1994), a tax-cut cannot be self-financing in this general model under the plausible parameter values since the growth rate effect is smaller than in his one-sector AK model. However, the tax rate which generates the greatest amount of stream of government expenditures is not high, compared to Pecorino (1995)'s revenue-maximizing tax rate, in spite of using his parameter values. It is because Pecorino (1995) does not take account of the transition path, and therefore does not capture the effect of the change in the interest rate for evaluating the present value.

Figure 5A to 5P show the time paths of several main variables along the transition path when the tax rate is raised from 40% to 46%, where time $-1$ means the old steady state and time 0 means the instant when the tax rate is changed. During a transition, interest rates can change, and therefore these effects must be taken into account for evaluating the present value. In contrast, the tax revenue-human capital ratio and the tax revenue-GNP ratio are stable. Note that physical capital intensity in sector 1 is always greater than that in sector 2 along the transition path.

V. Sensitivity Analysis

Table 4 reports results of the sensitivity analysis with respect to parameters $\sigma$ and $\Omega$. The computational results turn out to depend crucially on these parameter values.

When $\sigma = 4$ (i.e., the elasticity of intertemporal substitution $= 0.25$), the growth rate effect is smaller. For example, when the tax rate is raised from 40% to 45%, the long-run growth rate changes from 1.5% to 1.32% (1.28% in the base case). As a result, the tax rate which maximizes $\Delta g/g$ and $\Delta g/G$, is 54% and 62%, respectively, and higher than in the base case. The welfare loss is equivalent to 9.3% and 17.1% decrease in consumption every period, respectively. In this case, the welfare loss is relatively small, compared to the base case. On the other hand, when $\sigma = 1.01$ (i.e., near the log utility case), the growth rate effect is relatively large. For example, when the tax rate is raised to 45%, the growth rate is lowered to 0.97%. As a result, we can no longer finance the original stream of government expenditures by raising the tax rate. In other words, a tax-cut is self-financing, and therefore the conclusion of Ireland (1994) holds. However, the size of the effect is much smaller compared to his result. The tax rate which maximizes $\Delta g/g$ is 34%, and we can obtain the welfare gain equivalent to 11.1% increase in consumption every period (about 40% increase in consumption in Ireland (1994)). The tax rate which maximizes $\Delta G/G$ is also less than 40%, i.e., 39%, and the resulting welfare gain is smaller, i.e., 1.9% increase in consumption every period. When the elasticity of intertemporal substitution is large, the economy adjusts more to a given amount of changes in the interest rate, and as a result, the growth rate effect is larger.

The parameter of labor supply elasticity, $\Omega$, also has a large effect on the growth rate. When $\Omega = 0.1$, the labor supply is fairly inelastic, and the growth rate effect is smaller. For example, when the tax rate is raised to 45%, the growth rate becomes 1.3%. The tax rate which maximizes $\Delta g/g$ and $\Delta G/G$, is 56% and 66%, respectively. The resulting welfare loss is equivalent to 8.9% and 17.8% decrease in consumption every period, respectively. When $\Omega = 2$, however, the growth rate effect is relatively large. As a result, the tax rate which maximizes $\Delta g/g$ is 39%, and the resulting welfare gain is equivalent to 1.3% increase in consumption permanently. If we cut the tax rate by 1 more percent, for example, we can obtain the welfare gain of 2.5% increase in annual consumption, while still satisfying the government’s present value budget constraint. Therefore, a tax-cut is self-financing in terms of the interpretation (b). In this case, however, a tax-cut is not self-financing in terms of the interpretation (c). The tax rate which maximizes $\Delta G/G$ is 45%, and the resulting welfare loss is equivalent to 6.9% decrease in consumption every period. We can no longer finance the original stream of a constant $G$ by cutting the tax rate in this case. Previous studies on the traditional static Laffer curve emphasized the disincentive effect of taxation on static labor supply decisions. In this model, in addition to the above effect, the disincentive effect on labor supply has the negative effect on the human capital accumulation, and therefore, the negative effect on the growth rate.

As Stokey and Rebelo (1995) point out, these two parameters have a large effect on the growth rate. As a result, whether a tax-cut is self-financing or not crucially depends on these parameters. Even in a two-sector endogenous growth model where a human capital accumulation is lightly taxed and thus the growth rate effect is small, the results can be overturned, depending on these parameter values. This large sensitivity

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79See footnote (17).

80Ireland (1994) assumes that $\sigma = 1$. When $\sigma = 1$, as shown above, his conclusion holds even in a two-sector endogenous growth model.
of the results with respect to these parameters in the sense that the conclusion of Ireland (1994) can hold under Pecorino’s parameter set of sensitivity analysis is not captured in Pecorino (1995), since he argues his turning point is always higher than Ireland’s even under his parameter set of sensitivity analysis. In summary, if the elasticity of intertemporal substitution is small and the labor supply is fairly inelastic (these cases are considered to be empirically more relevant), the economy is on the upward-sloping part of the Laffer curve.

VI. Concluding Remarks

This paper analyzes the dynamic Laffer curve in an endogenous growth framework taking account of transition paths. It shows that instead of calculating the maximum point of the present value of tax revenues, an alternative measure is required to check a turning point of the Laffer curve in an endogenous growth framework. It then evaluates tax revenues from a viewpoint of (i) the constant stream of or (ii) the constantly growing stream of government expenditures that can be financed. A turning point of the Laffer curve is then given by the tax rate which maximizes such streams of government expenditures that can be financed.

The conflicting different results in the existing literature mainly come from the fact that existing papers use different concepts of the Laffer curve and different key parameter values. This paper examines these issues in a unified framework with a unified measure, interprets the computational results in terms of government expenditures that can be financed, and shows that, in a unified framework with a unified measure, differences are not as great as Pecorino (1995) argues, and the results crucially depend on the assumed parameter values.

If the elasticity of intertemporal substitution is small and the labor supply is fairly inelastic (this case is considered to be empirically more relevant), a tax-cut is not self-financing. Under the parameter set of Pecorino (1995), the turning point of the Laffer curve occurs at 46% in terms of the criterion (ii), and at 53% in terms of (i), and is lower than Pecorino’s turning point under his parameter set. It occurs at an even higher rate if these two elasticities are smaller. Therefore, the economy is on the upward-sloping part of the Laffer curve. However, if the elasticity of intertemporal substitution is near or greater than 1, or if the labor supply is fairly elastic, a tax-cut can be self-financing, and then we can obtain a welfare gain, satisfying the present value budget constraint even in a two sector endogenous growth model where a human capital accumulation is lightly taxed. Even in that case, however, the resulting welfare gain is much smaller than Ireland (1994) finds.

Further, if we can incorporate tax avoidance behavior into our model, the tax base shrinking effect may be bigger. This effect is not captured in this paper, and it might be interesting to pursue. For example, Feldstein (1995) shows that the welfare cost of taxation becomes substantially higher when we take account of the tax avoidance behavior. He shows that even if the labor supply elasticity is very small, the welfare cost of taxation may be much higher than previous studies find. A similar effect might apply to our model. This topic may be pursued in a future research.
Appendix

A Laffer Curve in the Ramsey-Cass-Koopmans Model

Consider the following simple Ramsey-Cass-Koopmans model. Technology is described by the Cobb-Douglas production function \( y = f(k) = Ak^\alpha \), where \( y \) is an output-labor ratio, \( A \) a constant, \( k \) a physical capital-labor ratio, and \( \alpha \) the capital income share. Assuming for simplicity that the rate of technical progress, that of population growth, and that of depreciation are all 0, the following system of differential equations describes the motion of the economy:

\[
\begin{align*}
(A.1) \quad \dot{k} &= f(k) - c = Ak^\alpha - c, \\
(A.2) \quad \dot{c} &= \frac{c}{\sigma}[(1 - \tau)\alpha Ak^{\alpha-1} - \rho],
\end{align*}
\]

where \( \tau \) is an income tax rate, \( c \) the consumption per capita, \( \sigma \) the inverse of the elasticity of intertemporal substitution, \( \rho \) the rate of time preference, and \( \gamma \) the after-tax interest rate that is endogenously determined by this equation. Tax revenues are assumed to be rebated in a lump sum manner. Linearizing this system at the steady state, then we can obtain

\[
(A.3) \quad \frac{d}{dt}[k - k_s c - c_s] = \begin{bmatrix} \alpha A(k_s)^{\alpha-1} - 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} k - k_s \\ c - c_s \end{bmatrix},
\]

where * denotes the steady state value.

From (A.2), the following relationship holds in the steady state.

\[
(A.4) \quad \rho = \gamma = (1 - \tau) f'(k) = (1 - \tau)\alpha Ak^{\alpha-1}.
\]

Then, the steady state capital stock is

\[
(A.5) \quad k_s = \left(\frac{(1 - \tau)\alpha A}{\rho}\right)^{\frac{1}{1-\alpha}}.
\]

From (A.1), (A.2), (A.3), and (A.5), the convergence rate \( \kappa \) is calculated as the absolute value of the negative eigenvalue of the dynamic matrix of equation (A.3).

\[
(A.6) \quad \kappa = \frac{\rho}{1 - \tau} \left(\sqrt{1 + 4(1 - \tau)^2 \frac{1-\alpha}{\alpha} - 1}\right).
\]

It can be verified that \( \kappa \) is increasing in \( \tau \).

Tax revenue in the steady state for each period is

\[
(A.7) \quad TR = \tau f(k) = \tau Ak^\alpha = \tau A \left(\frac{(1 - \tau)\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}} = C\tau(1 - \tau)^{\frac{\alpha}{1-\alpha}} \equiv TR(\tau),
\]

where \( C \) is a constant. Thus, Therefore, from a viewpoint of comparing tax revenues across steady states, \( TR \) is maximized at \( \tau^* = 1 - \alpha \), increasing for \( 0 < \tau < \tau^* \), and decreasing for \( \tau > \tau^* \). In addition, \( TR'' < 0 \) can be verified at least if \( \alpha < 0.5 \). Therefore, \( TR \) is strictly concave.

Raise the tax rate from \( \tau \) to \( \tau + \Delta \tau \) (\( \Delta \tau > 0 \)). Then, for a given initial tax rate \( \tau \),

\[
(A.9) \quad \overline{TR} = \tau A \left(\frac{(1 - \tau)\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}} = C\tau(1 - \tau)^{\frac{\alpha}{1-\alpha}}.
\]

Thus, \( \overline{TR} \) is constant for a given \( \tau \).

\[
(A.10) \quad TR^* = (\tau + \Delta \tau) A \left(\frac{(1 - \tau - \Delta \tau)\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}} = C(\tau + \Delta \tau)(1 - \tau - \Delta \tau)^{\frac{\alpha}{1-\alpha}}.
\]
Assume that \( \tau < 1 - \alpha \). From the above argument, \( TR^* \) is strictly concave, increases if \( \tau + \Delta \tau < 1 - \alpha \), is maximized at \( \tau + \Delta \tau = 1 - \alpha \), and decreases if \( \tau + \Delta \tau > 1 - \alpha \) for a given \( \tau \).

\[
(A.11) \quad TR_0 = (\tau + \Delta \tau)A \left( \frac{(1 - \tau)A}{\rho} \right)^{\frac{\alpha}{\alpha - 1}} = C(\tau + \Delta \tau)(1 - \tau)^{\frac{\alpha}{\alpha - 1}}.
\]

Equation (A.11) holds since the capital stock is fixed at time 0 when the tax rate is changed. Thus, \( TR_0 \) is linearly increasing in \( \Delta \tau \). Chamley (1985) shows that \( WR \) is expressed as

\[
(A.12) \quad WR = \frac{\rho}{\rho + \kappa} TR_0 + \frac{\kappa}{\rho + \kappa} TR^* - \overline{TR}.
\]

Then, clearly from the above argument, \( WR \) increases at first. However, if the tax rate \( \tau + \Delta \tau \) exceeds \( 1 - \alpha \), \( TR^* \) decreases (in addition, note that \( TR_0 \) is linear, while \( TR^* \) is not) and \( \kappa \) increases (so that the weight in \( WR \) shifts from \( TR_0 \) to \( TR^* \)), and therefore \( WR \) may decrease beyond some tax rate.

Since the above argument is based on the linear approximation, this result is best applied in the case where \( \tau \) is very near but less than \( 1 - \alpha \), and \( \Delta \tau \) is small. However, the similar global argument is possible by numerical simulations.

### B Growth Rate Function and a Turning Point

First, consider the case where \( \sigma = 1 \) and the initial tax base \( AK_0 \) is constant. Assume that the growth rate function \( \nu(\tau) \) is differentiable. Since \( \gamma_d - \nu(\tau) + \tau \nu'(\tau) = \rho + \nu(0) - \nu(\tau) + \tau \nu'(\tau) \) holds,

\[
\text{sgn}\{PVR^P(\tau)\} = \text{sgn}\{\rho + \nu(0) - \nu(\tau) + \tau \nu'(\tau)\}.
\]

When \( \tau = 0 \), \( \text{sgn}\{\rho + \nu(0) - \nu(\tau) + \tau \nu'(\tau)\} = \text{sgn}\{\rho\} > 0 \). Thus, \( PVR^P(\tau) \) is upward sloping at least when \( \tau \) is very small, by continuity of \( PVR^P(\tau) \) function.

#### B.1. \( \nu(\tau) \) is linear

When \( \nu(\tau) \) is linear, then \( \nu(0) = \nu(\tau) - \nu'(\tau)\tau \) for all \( \tau \). Thus, \( \text{sgn}\{\rho + \nu(0) - \nu(\tau) + \tau \nu'(\tau)\} = \text{sgn}\{\rho\} > 0 \) for all \( \tau \). Therefore, \( PVR^P(\tau) \) is always upward sloping, and thus there is no turning point.

![Growth Rate Function](image)

#### B.2. \( \nu(\tau) \) is strictly concave

When \( \nu(\tau) \) is strictly concave, then \( \nu(0) < \nu(\tau) - \nu'(\tau)\tau \) for all \( \tau \). Thus, \( \nu(0) - \nu(\tau) + \nu'(\tau)\tau < 0 \). If \( \nu(0) - \nu(\tau) + \nu'(\tau)\tau \) is negative enough (i.e., \( \nu(0) - \nu(\tau) + \nu'(\tau)\tau < -\rho < 0 \)) so that \( \rho + \nu(0) - \nu(\tau) + \nu'(\tau)\tau < 0 \).

\[\footnote{See figure 2A.}\]
for some \( \tau^* \), then, clearly, \( PVR^P(\tau) \) is downward sloping beyond \( \tau^* \).\(^{82}\)

\[ \frac{d}{d\tau}[\nu(0) - \nu(\tau) + \tau\nu'(\tau)] = \tau\nu''(\tau) < 0. \]

\[ PVR^P(\tau) = \frac{TB(1 + \tau TB/\gamma_d - \nu(\tau)) + \tau\nu'}{\gamma_d - \nu(\tau)} = \frac{TB(1 - \xi(\tau))(\gamma_d - \nu(\tau)) + \tau\nu}{\gamma_d - \nu(\tau)} \]

\[ \text{sgn}\{PVR^P(\tau)\} = \text{sgn}\{(1 - \xi(\tau))(\rho + \nu(0) - \nu(\tau)) + \tau\nu'(\tau)\} \]

\[ = \text{sgn}\{(1 - \xi(\tau))\rho - \xi(\tau)[\nu(0) - \nu(\tau)] + [\nu(0) - \nu(\tau) + \tau\nu'(\tau)]\}. \]

\( B.2.1. \) changing initial tax base

In the more general case, initial tax base \( AK_0 \) may change, depending on \( \tau \). Let the initial tax base be \( TB(\tau) \). For example, Pecorino (1995)’s initial tax base is the initial tax base at the steady state normalized such that potential GNP evaluated at prices in a tax-free economy is 1. Then, Pecorino’s \( TB(\tau) \) is expressed as follows.

Clearly from the above figure, \( TB(\tau) < 0 \) and \( TB'(\tau) < 0 \) holds. As the tax rate becomes high, human capital devoted to production is smaller and intersectoral allocations of both capital are distorted, and therefore, the initial tax base is smaller and smaller. Define the elasticity \( \xi(\tau) \equiv -\frac{TB'(\tau)}{TB(\tau)} \), where \( \xi(\tau) > 0 \) since \( TB'(\tau) < 0 \) and \( TB''(\tau) < 0 \). Since \( PVR^P(\tau) = \frac{TB(\tau)}{\gamma_d - \nu(\tau)} \), we have by differentiation

\[ PVR^P(\tau) = \frac{TB(1 + \tau TB/\gamma_d - \nu(\tau)) + \tau\nu'}{\gamma_d - \nu(\tau)} \]

Thus,

\[ \text{sgn}\{PVR^P(\tau)\} = \text{sgn}\{(1 - \xi(\tau))(\rho + \nu(0) - \nu(\tau)) + \tau\nu'(\tau)\} \]

\[ = \text{sgn}\{(1 - \xi(\tau))\rho - \xi(\tau)[\nu(0) - \nu(\tau)] + [\nu(0) - \nu(\tau) + \tau\nu'(\tau)]\}. \]

\(^{82}\)Note that \( \frac{d}{d\tau}[\nu(0) - \nu(\tau) + \nu'(\tau)\tau] = \tau\nu''(\tau) < 0. \)
Both the second and third term are always negative, while the first term may be positive. However, as $\tau$ becomes high, $1 - \xi(\tau)$ becomes smaller and smaller, and therefore $PVR^P(\tau)$ is negative beyond some $\tau$.

**B.2.2. when $\sigma \neq 1$**

In this case, $\gamma_d = \rho + \sigma \nu(0)$ holds. Then,

$$
\text{sgn}\{PVR^P(\tau)\} = \text{sgn}\{(1 - \xi(\tau))[\rho + (\sigma - 1)\nu(0)]
- \xi(\tau)\nu(0) - \nu(\tau) + [\nu(0) - \nu(\tau) + \tau\nu'(\tau)]\}.
$$

By the same argument as above, $PVR^P(\tau)$ can be negative beyond some $\tau$.

**C Parameter Values and Calibration**

**C.1. Parameter Values**

Following parameter values are assumed to be the same as those of Pecorino (1995).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>government expenditures on goods and services as a percentage of revenues</td>
<td>$\eta$</td>
</tr>
<tr>
<td>physical capital share parameter in sector 1</td>
<td>$\alpha_K^1$</td>
</tr>
<tr>
<td>physical capital share parameter in sector 2</td>
<td>$\alpha_K^2$</td>
</tr>
<tr>
<td>elasticity of substitution between physical capital and human capital in sector 1</td>
<td>$1/(1 - \psi_1)$</td>
</tr>
<tr>
<td>elasticity of substitution between physical capital and human capital in sector 2</td>
<td>$1/(1 - \psi_2)$</td>
</tr>
<tr>
<td>physical capital depreciation rate</td>
<td>$\delta_K$</td>
</tr>
<tr>
<td>human capital depreciation rate</td>
<td>$\delta_H$</td>
</tr>
<tr>
<td>intertemporal elasticity of substitution</td>
<td>$1/\sigma$</td>
</tr>
<tr>
<td>leisure share parameter</td>
<td>$\Omega$</td>
</tr>
<tr>
<td>rate of time preference</td>
<td>$\rho$</td>
</tr>
<tr>
<td>wage tax rate in sector 1</td>
<td>$t^1_H$</td>
</tr>
<tr>
<td>wage tax rate in sector 2</td>
<td>$t^2_H$</td>
</tr>
<tr>
<td>physical capital income tax rate in sector 1</td>
<td>$t^1_K$</td>
</tr>
<tr>
<td>physical capital income tax rate in sector 2</td>
<td>$t^2_K$</td>
</tr>
<tr>
<td>consumption tax rate</td>
<td>$t_c$</td>
</tr>
</tbody>
</table>

Scale parameters $A^1$ and $A^2$ are computed by the calibration procedure in the same way as in Pecorino (1995). That is, $A^1$ and $A^2$ are set such that the original steady state growth rate is equal to 1.5% under the condition that $A^1 = A^2$.  

| Scale parameter in sector 1 | $A^1$ | 0.26 |
| Scale parameter in sector 2 | $A^2$ | 0.26 |

**C.2. Calibration Procedure**

Calibration procedure is the same as that of Pecorino (1995). That is, $A^1$ and $A^2$ are set such that the following condition holds under the condition that $A^1 = A^2$:

$$
0.015 = \left[\frac{1 + (1 - t^1_K)(r^1 - \delta_K)}{1 + \rho}\right]^\frac{1}{3} - 1.
$$

In the sensitivity analysis, these parameters are recalibrated in the same way.

---

83The condition that $A^1 = A^2$ is not stated explicitly in Pecorino (1995). However, unless we impose this condition, we cannot determine $A^1$ and $A^2$ uniquely. Actually, Pecorino (1993) states this condition explicitly.
D Computation of the matrix $M$

Appendix D describes the method to compute the matrix $M$. First, partially differentiate the function $\Gamma$ with respect to $X_{t+1}$, and evaluate the derivatives at the steady state values.

\[
\frac{\partial \Gamma^1}{\partial a_{t+1}} = (m + (1 - \delta_H)),
\]
\[
\frac{\partial \Gamma^1}{\partial b_{t+1}} = 0,
\]
\[
\frac{\partial \Gamma^1}{\partial p_{t+1}^2} = 0,
\]
\[
\frac{\partial \Gamma^2}{\partial a_{t+1}} = (1 - t_H^1)p^2[(1 - l)\frac{\partial w^1}{\partial a} - w^1 \frac{\partial l}{\partial a}] - (1 - t_K^1)p^2 \frac{\partial r^1}{\partial a},
\]
\[
\frac{\partial \Gamma^2}{\partial b_{t+1}} = (1 - t_H^1)p^2[(1 - l)\frac{\partial w^1}{\partial b} - w^1 \frac{\partial l}{\partial b}] - (1 - t_K^1)p^2 \frac{\partial r^1}{\partial b},
\]
\[
\frac{\partial \Gamma^2}{\partial p_{t+1}^2} = [(1 - t_H^1)w^1(1 - l) + (1 - \delta_H)] + (1 - t_H^1)p^2[(1 - l)\frac{\partial w^1}{\partial p^2} - w^1 \frac{\partial l}{\partial p^2}] - (1 - t_K^1)p^2 \frac{\partial r^1}{\partial p^2},
\]
\[
\frac{\partial \Gamma^3}{\partial a_{t+1}} = (1 + (1 - t_K^1)(r^1 - \delta_K))\Omega(1 - \sigma) \frac{1}{b} \frac{\partial l}{\partial a} + (1 - t_K^1) \frac{\partial r^1}{\partial a},
\]
\[
\frac{\partial \Gamma^3}{\partial b_{t+1}} = (1 + (1 - t_K^1)(r^1 - \delta_K))\Omega(1 - \sigma) \frac{1}{b} \frac{\partial l}{\partial b} + (1 - t_K^1) \frac{\partial r^1}{\partial b},
\]
\[
\frac{\partial \Gamma^3}{\partial p_{t+1}^2} = (1 + (1 - t_K^1)(r^1 - \delta_K))\Omega(1 - \sigma) \frac{1}{b} \frac{\partial l}{\partial p^2} + (1 - t_K^1) \frac{\partial r^1}{\partial p^2}.
\]

Next, partially differentiate the function $\Gamma$ with respect to $X_t$, and evaluate the derivatives at the steady state values.

\[
\frac{\partial \Gamma^1}{\partial a_t} = a \frac{\partial m}{\partial a} - (1 + (1 - t_K^1)(r^1 - \delta_K)) - (1 - t_K^1)a \frac{\partial r^1}{\partial a}
\]
\[-(1 - t_H^1)p^2[(1 - l)\frac{\partial w^1}{\partial a} - w^1 \frac{\partial l}{\partial a}] - (1 - \eta) \frac{\partial m}{\partial a} + p \frac{\partial m}{\partial a},
\]
\[
\frac{\partial \Gamma^1}{\partial b_t} = a \frac{\partial m}{\partial b} - (1 - t_K^1)a \frac{\partial r^1}{\partial b} - (1 - t_H^1)p^2[(1 - l)\frac{\partial w^1}{\partial b} - w^1 \frac{\partial l}{\partial b}]
\]
\[-(1 - \eta) \frac{\partial m}{\partial b} + (1 + t_c) + p \frac{\partial m}{\partial a},
\]
\[
\frac{\partial \Gamma^1}{\partial p_t^2} = a \frac{\partial m}{\partial p^2} - (1 - t_K^1)a \frac{\partial r^1}{\partial p^2} - (1 - t_H^1)w^1(1 - l)
\]
\[-(1 - t_H^1)p^2[(1 - l)\frac{\partial w^1}{\partial p^2} - w^1 \frac{\partial l}{\partial p^2}] - (1 - \eta) \frac{\partial m}{\partial p^2} + p \frac{\partial m}{\partial p^2} + m,
\]
\[
\frac{\partial \Gamma^2}{\partial a_t} = 0,
\]
\[
\frac{\partial \Gamma^2}{\partial b_t} = 0,
\]
\[
\frac{\partial \Gamma^2}{\partial p_t^2} = -(1 + (1 - t_K^1)(r^1 - \delta_K)).
\]

\^4This method follows Laitmer (1995).
\[
\frac{\partial \Gamma^3}{\partial a_t} = (1 + (1 - t_K^1)(r^1 - \delta_K))\Omega(1 - \sigma)(-\frac{1}{T} \frac{\partial l}{\partial a} - (1 + \rho)\sigma(m + (1 - \delta_H)))^{\sigma-1} \frac{\partial m}{\partial a},
\]
\[
\frac{\partial \Gamma^3}{\partial b_t} = (1 + (1 - t_K^1)(r^1 - \delta_K))((-\sigma)(-\frac{1}{b^1} + \Omega(1 - \sigma)(-\frac{1}{T} \frac{\partial l}{\partial b} - (1 + \rho)\sigma(m + (1 - \delta_H)))^{\sigma-1} \frac{\partial m}{\partial b},
\]
\[
\frac{\partial \Gamma^3}{\partial p_t^2} = (1 + (1 - t_K^1)(r^1 - \delta_K))\Omega(1 - \sigma)(-\frac{1}{T} \frac{\partial l}{\partial p^2} - (1 + \rho)\sigma(m + (1 - \delta_H)))^{\sigma-1} \frac{\partial m}{\partial p^2}.
\]

From above computations, if we can get the derivatives such as, for example, \(\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial f}{\partial p^2}\) and so on, the derivatives \(\frac{\partial f(X, X^*)}{\partial X_{i+1}}\) and \(\frac{\partial f(X, X^*)}{\partial X_i}\) can be computed, respectively. Then we can get the matrix
\[
M = \left[\frac{\partial f(X, X^*)}{\partial X_{i+1}}\right]^{-1} \frac{\partial f(X, X^*)}{\partial X_t}.
\]

Actually, such derivatives can be easily computed by using the Implicit Function Theorem. Consider equations (41), (42), (43), (44), (45), (46), (47), (48), (49), (50), (54), (55), and (56) in Section III.E. Totally differentiate these 13 equations and evaluate them at the steady state values.\(^{85}\)

(D.1) \(dy^1 = (f^1) \left( \frac{\phi^1}{u^1} da + \frac{a}{u^1} d\phi^1 - \frac{\phi^1 a}{(u^1)^2} du^1 \right),\)

(D.2) \(dy^2 = (f^2) \left( \frac{\phi^2}{u^2} da + \frac{a}{u^2} d\phi^2 - \frac{\phi^2 a}{(u^2)^2} du^2 \right),\)

(D.3) \(d\phi^1 + d\phi^2 = 0,\)

(D.4) \(du^1 + du^2 + dl = 0,\)

(D.5) \((1 - t_K^1)dr^1 = (1 - t_K^2)dr^2,\)

(D.6) \((1 - t_K^1)du^1 = (1 - t_K^2)du^2,\)

(D.7) \(dr^1 = (f^{1n}) \left( \frac{\phi^1}{u^1} da + \frac{a}{u^1} d\phi^1 - \frac{\phi^1 a}{(u^1)^2} du^1 \right),\)

(D.8) \(dr^2 = f^{2n} dp^2 + p^2 (f^{2n}) \left( \frac{\phi^2}{u^2} da + \frac{a}{u^2} d\phi^2 - \frac{\phi^2 a}{(u^2)^2} du^2 \right),\)

(D.9) \(p^2 dw^1 + w^1 dp^2 = - \left( \frac{\phi^1}{u^1} a \right) \left( f^{1n} \right) \left( \frac{\phi^1}{u^1} da + \frac{a}{u^1} d\phi^1 - \frac{\phi^1 a}{(u^1)^2} du^1 \right),\)

(D.10) \(dw^2 = - \left( \frac{\phi^2}{u^2} a \right) \left( f^{2n} \right) \left( \frac{\phi^2}{u^2} da + \frac{a}{u^2} d\phi^2 - \frac{\phi^2 a}{(u^2)^2} du^2 \right),\)

(D.11) \(\Omega db = (1 - t_H^1) \left[ w^1 p^2 dl + w^1 dp^2 + p^2 dw^1 \right],\)

(D.12) \(dn = t_K(r^1 - \delta_K)[\phi^1 da + ad\phi^1] + t_K^1 \phi^1adr^1 + t_K^2 (r^2 - \delta_K)[\phi^2 da + ad\phi^2] + t_K^2 \phi^2 adr^2 + t_H^1 [u^1 w^1 dp^2 + u^1 p^2 dw^1 + w^1 p^2 da^1] + t_H^1 [u^2 w^2 dp^2 + u^2 p^2 dw^2 + w^2 p^2 da^2] + t_e db,\)

(D.13) \(dm = u^2 dy^2 + y^2 du^2,\)

where
\[
f^j(k_j) = A^j [a^j K_{k_j}^{\psi_j} + (1 - a^j K_{k_j}^{\psi_j})]^{\frac{1}{\psi_j}},
\]
\[
f^j(k_j) = A^j [a^j K_{k_j}^{\psi_j} + (1 - a^j K_{k_j}^{\psi_j})]^{\frac{1}{\psi_j}} - 1,
\]
\[
f^{jn}(k_j) = (\psi_j - 1) A^j [a^j K_{k_j}^{\psi_j} + (1 - a^j K_{k_j}^{\psi_j})]^{\frac{1}{\psi_j} - 1} + (1 - \psi_j) A^j [a^j K_{k_j}^{\psi_j} + (1 - a^j K_{k_j}^{\psi_j})]^{\frac{1}{\psi_j} - 2}.
\]

By setting \(db_t = dp_t^2 = 0\) in the above system of simultaneous equations (D.1) to (D.13), we can get the partial derivatives with respect to \(a\) evaluated at the steady state such as, \(\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial f}{\partial p^2}\), etc. Similarly, we can get \(\frac{\partial g}{\partial m}, \frac{\partial g}{\partial b}, \frac{\partial g}{\partial p^2}\), etc. by setting \(da_t = db_t = 0\).

\(^{85}\)For example, \((u^2)^2\) denotes the square of the variable \(u^2\), the portion of time devoted to the production in sector 2, although the notation may be confusing.
E Computational Procedure

Compute the matrix \( M \) in the way described in Appendix D. Then, we can compute eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) and eigenvectors \( v_1, v_2, v_3 \) of the matrix \( M \).

If one eigenvalue, say \( \lambda_1 \), has modulus less than one and other two (i.e. \( \lambda_2, \lambda_3 \)) have modulus greater than one, for any \( H_0 \) and \( K_0 \) there exists one and only one equilibrium path converging to the steady state.

Let \( v_1 \) be the eigenvector corresponding to the eigenvalue \( \lambda_1 \), and define \( v_1 \equiv \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} \). For convenience, normalize \( v_{13} \) to 1.

As described in Section III.E., the equations (41) to (56) and the transversality conditions describe the motion of this economy. On the other hand, from equations (41') to (56'), we can get the steady state solutions of these equations,

\[
a^*, b^*, p^{2*}, y^{1*}, y^{2*}, \phi^{1*}, \phi^{2*}, u^{1*}, u^{2*}, l^*, r^{1*}, r^{2*}, w^{1*}, w^{2*}, m^*, and n^*.
\]

Shoot backwards from the steady state in the opposite direction of \( v_1 \). Specifically, define \( a^{**} \equiv a^* - \varepsilon \cdot v_{11}, b^{**} \equiv b^* - \varepsilon \cdot v_{12}, p^{2**} \equiv p^{2*} - \varepsilon \cdot v_{13} \), where \( \varepsilon \) is a very small value.\(^{86}\) Given \( a^{**}, b^{**}, \) and \( p^{2**} \), we can get \( y^{1**}, y^{2**}, \phi^{1**}, \phi^{2**}, u^{1**}, u^{2**}, l^{**}, r^{1**}, r^{2**}, w^{1**}, w^{2**}, m^{**}, \) and \( n^{**} \) by solving 13 equations (41) to (50) and (54) to (56). In equations (41) to (56), set \( a_{i+1} = a^{**}, b_{i+1} = b^{**}, p_{i+1}^{2**} = p^{2**}, y_{i+1}^{1**} = y^{1**}, y_{i+1}^{2**} = y^{2**}, \phi_{i+1}^{1**} = \phi^{1**}, \phi_{i+1}^{2**} = \phi^{2**}, u_{i+1}^{1**} = u^{1**}, u_{i+1}^{2**} = u^{2**}, l_{i+1}^{**} = l^{**}, r_{i+1}^{1**} = r^{1**}, r_{i+1}^{2**} = r^{2**}, w_{i+1}^{1**} = w^{1**}, w_{i+1}^{2**} = w^{2**}, m_{i+1} = m^{**}, \) and \( n_{i+1} = n^{**} \), then we can get the following 16 variables \( a_1, b_1, p_1^{2*}, y_1^{1*}, y_1^{2*}, \phi_1^{1*}, \phi_1^{2*}, u_1^{1*}, u_1^{2*}, l_1, r_1^{1}, r_1^{2}, w_1^{1}, w_1^{2}, m_1, \) and \( n_1 \) by solving equations (41) to (56). Given these \( t \) period variables, we can also get the \( t-1 \) period variables \( a_{i-1}, b_{i-1}, p_{i-1}^{2*}, y_{i-1}^{1*}, y_{i-1}^{2*}, \phi_{i-1}^{1*}, \phi_{i-1}^{2*}, u_{i-1}^{1*}, u_{i-1}^{2*}, l_{i-1}, r_{i-1}^{1}, r_{i-1}^{2}, w_{i-1}^{1}, w_{i-1}^{2}, m_{i-1}, \) and \( n_{i-1} \) by solving equations (41) to (56). Continue this loop until a coincides with the historically given value \( a_0 = \frac{K_0}{M_0} \). Since this is a discrete system, a does not always coincide exactly with the historically given value \( a_0 = \frac{K_0}{M_0} \) for any \( \varepsilon \). Therefore we can adjust the size of \( \varepsilon \) so that \( a \) exactly coincides with \( a_0 = \frac{K_0}{M_0} \). This is called the Backward Shooting Method.\(^{87}\) From the above procedure, we can get the steady values of variables,

\[
\begin{align*}
a_0, a_1, & \ldots, a^{**}, \\
b_0, b_1, & \ldots, b^{**}, \\
p_{0}^{2}, p_{1}^{2*}, & \ldots, p^{2**}, \\
\vdots, \\
n_0, n_1, & \ldots, n^{**}.
\end{align*}
\]

After reaching \( a^{**}, b^{**}, p^{2**}, y^{1**}, y^{2**}, \phi^{1**}, \phi^{2**}, u^{1**}, u^{2**}, l^{**}, r^{1**}, r^{2**}, w^{1**}, w^{2**}, m^{**}, \) and \( n^{**} \), it is assumed for simplicity that these variables take the steady state values, i.e., \( a^*, b^*, p^{2*}, y^{1*}, y^{2*}, \phi^{1*}, \phi^{2*}, u^{1*}, u^{2*}, l^*, r^{1*}, r^{2*}, w^{1*}, w^{2*}, m^*, \) and \( n^* \). Therefore we can get the whole stream of these variables,

\[
\begin{align*}
a_0, a_1, & \ldots, a^{**}, a^*, a^{*}, a^{**}, \\
b_0, b_1, & \ldots, b^{**}, b^*, b^{*}, b^{**}, \\
p_{0}^{2}, p_{1}^{2*}, & \ldots, p^{2**}, p^{2*}, p^{2**}, \\
\vdots, \\
n_0, n_1, & \ldots, n^{**}, n^*, n^{*}, \ldots, n^{**}.
\end{align*}
\]

In this way, we can get the whole time paths of these variables.

\(^{86}\)This rule applies only to the case where the initial value of \( a \) is smaller than the steady state value of \( a \). In the case where the initial value of \( a \) is greater than the steady state value of \( a \), define \( a^{**} = a^* + \varepsilon \cdot v_{11}, b^{**} = b^* + \varepsilon \cdot v_{12}, p^{2**} = p^{2*} + \varepsilon \cdot v_{13} \).

\(^{87}\)The Time Elimination Method in Mulligan and Martin (1991, 1993) is basically based on the idea of the Backward Shooting Method, and in addition eliminates time from differential equations. That is, just as “one key advantage of the Time-Elimination Method is that it transforms the boundary value problem described by the equations of motion and the TVC’s into an initial value problem” (Mulligan and Martin (1991)), the Backward Shooting Method also makes such a transformation. The Time-Elimination Method is easier to handle, while the Backward Shooting Method is more widely applicable. By eliminating time, the Time Elimination Method makes the computational program easy and saves the computation time dramatically, but other variables such as \( r \) and \( w \) must be expressed as explicit functions of basic variables. On the other hand, in the Backward Shooting Method, the computational program is a little bit more difficult and the computation time is longer, but other variables such as \( r \) and \( w \) do not have to be expressed as explicit functions of basic variables. They have only to be expressed as implicit functions of basic variables. This is one advantage of this method. Since the model in this paper is complicated, it turns out that we cannot apply the Time-Elimination Method, but can do the Backward Shooting Method in order to solve this model.
Next, consider the equation (33), i.e., \( \frac{H_{t+1}}{H_t} = (1 - \delta_H) + m_t \). Since we know \( H_0 \) from the history and get the whole time path of \( m_t \) from above computations, we can compute the whole path of \( H_t \). From the path of \( H_t \), we can obtain the whole time path of \( m_t \) from above computations, we can compute the whole path of \( H_t \). From the path of \( H_t \), we can obtain the whole time paths of \( K_t, c_t, Y^1_t, Y^2_t, T^H_t \), and \( TR_t \) by multiplying the path of \( H_t \) by paths of \( a_t, b_t, y^1_t, y^2_t, m_t, \) and \( n_t \).

Let \( T \) be the length of the period during which the economy reaches the steady state.\(^{88} \) Then, the path of human capital, the present value of tax revenues, the present value of government expenditures, and the welfare change are computed as follows.

### E.1. Human Capital

First, consider the human capital accumulation equation (33).

(E.1) \[
H_{t+1} = [(1 - \delta_H) + m_t]H_t,
\]

Since we know the time path of \( m_t \), \( H_t \) can be computed as follows:

(E.2) \[
H_t = H_0 \prod_{s=0}^{t-1} [(1 - \delta_H) + m_s].
\]

If the economy continues to stay at the original steady state from the beginning, then

(E.3) \[
H_t = H_0(1 - \delta_H) + \bar{m}]^t,
\]

where \( \bar{m} \) is a constant value of \( m \) at the original steady state.

Suppose that as a result of the change in the tax rate the economy is on the transition path toward the new steady state from the period 0 to the period \( T \), reaches the new steady state at the period \( T + 1 \), and continues to stay at the new steady state from the period \( T + 1 \) on. Then \( H_t \) can be expressed as follows:

(E.4) \[
H_t = H_0 \prod_{s=0}^{T-1} [(1 - \delta_H) + m_s]
\]

if \( 0 \leq t \leq T \),

\[
= \{H_0 \prod_{s=0}^{T-1} [(1 - \delta_H) + m_s]\}[(1 - \delta_H) + m^*]^{(T+1)}
\]

if \( t \geq T + 1 \),

where \( m^* \) is a constant value of \( m \) at the new steady state.

### E.2. Present Value of Tax Revenues (PVR)

Define \( \gamma_t \) as an after-tax rate of return net of depreciation. That is, \( \gamma_t = (1 - t_K)(r^1_t - \delta_K) \). Then, the present value of tax revenues discounted by the after-tax rate of interest net of depreciation is expressed as follows:

(E.5) \[
PVR = \sum_{t=0}^{\infty} \frac{TR_t}{\prod_{s=1}^{t}(1 + \gamma_s)}.
\]

The \( PVR \) in the case where the economy continues to stay at the original steady state is expressed as follows:

(E.6) \[
PVR = \sum_{t=0}^{\infty} \frac{\pi H_t}{\prod_{s=1}^{t}(1 + \gamma)} = \pi H_0 \sum_{t=0}^{\infty} \frac{[(1 - \delta_H) + \bar{m}]^t}{(1 + \gamma)^t} = \pi H_0 \frac{1}{1 - (1 - \delta_H + \bar{m})},
\]

where \( - \) denotes the old steady state value. On the other hand, the \( PVR \), in the case where the economy follows the transition path and reaches the new steady state after the change in tax rates, is expressed as

\(^{88} \) Then, for example, \( a^{**} = a_T \).
where Ξ follows:

(E.7) \[ PVR^* = \sum_{t=0}^{T} \frac{n_t H_t}{\prod_{s=1}^{T-1} (1 + \gamma_s)} + \sum_{t=T+1}^{\infty} \frac{n_t H_t^*}{\prod_{s=1}^{t-1} (1 + \gamma_s)} \]

\[ = \sum_{t=0}^{T} \frac{n_t H_0 \prod_{s=0}^{t-1} [(1 - \delta_H) + m_s]}{\prod_{s=1}^{t} (1 + \gamma_s)} \]

\[ + \sum_{t=T+1}^{\infty} \frac{n_t H_0 \prod_{s=0}^{t-1} [(1 - \delta_H) + m_s][1 - \delta_H + m^*]^{t-(T+1)}}{\prod_{s=1}^{T} (1 + \gamma_s)} \]

\[ = H_0 \sum_{t=0}^{T} \frac{n_t \prod_{s=0}^{t-1} [(1 - \delta_H) + m_s]}{\prod_{s=1}^{T} (1 + \gamma_s)} + H_0 \frac{n_t \prod_{s=0}^{T} [(1 - \delta_H) + m_s]}{\prod_{s=1}^{T} (1 + \gamma_s)} \]

where Ψ is a discount factor and Ψ = \( \prod_{s=1}^{T} (1 + \gamma_s) \). Therefore, we can obtain

\[ \sum_{t=0}^{\infty} \frac{1}{(1 + \gamma)^t} = \frac{1}{1 - \frac{1}{1 + \gamma}}, \]

where \( \gamma^* \) is the after-tax rate of return at the new steady state, \( \hat{\gamma} \) the stream of after-tax rates of return when the tax rate is changed, and \((\{\gamma_t\}_{t=0}^{\infty}) = (\{\gamma_t\}_{t=0}^{\infty}, \{\gamma^*_t\}_{t=T+1}^{\infty}) \) holds.

### E.3. Present Value of Government Expenditures (PVG)

The present value of the original stream of government expenditures evaluated by the new after-tax interest rate net of depreciation is expressed as follows:

(E.8) \[ PVG^* = \sum_{t=0}^{T} \frac{\pi H_t}{\prod_{s=1}^{T} (1 + \gamma_s)} = \pi H_0 \sum_{t=0}^{T} \frac{[1 - \delta_H + \mu]^t}{\prod_{s=1}^{T} (1 + \gamma_s)} \]

\[ = \sum_{t=0}^{T} \frac{\pi H_0 [1 - \delta_H + \mu]^t}{\prod_{s=1}^{T} (1 + \gamma_s)} + \sum_{t=T+1}^{\infty} \frac{\pi H_0 [1 - \delta_H + \mu]^T [1 - \delta_H + \mu]^{T-t}}{\prod_{s=1}^{T} (1 + \gamma_s)} \]

\[ = \pi H_0 \sum_{t=0}^{T} \frac{[1 - \delta_H + \mu]^t}{\prod_{s=1}^{T} (1 + \gamma_s)} + \pi H_0 \frac{[1 - \delta_H + \mu]^{T+1}}{\prod_{s=1}^{T} (1 + \gamma_s)} \]

Therefore, from the government’s present value budget constraint, we have

(E.9) \[ PVR^* - PVG^* = \sum_{t=0}^{\infty} \frac{\Delta g H_0 [1 - \delta_H + \mu]^t}{\prod_{s=1}^{T} (1 + \gamma_s)} = \Delta g \cdot \Psi, \]

where Ψ is a discount factor and Ψ = \( \sum_{t=0}^{\infty} \frac{H_0 [1 - \delta_H + \mu]^t}{\prod_{s=1}^{T} (1 + \gamma_s)} = \frac{PVG^*}{\pi} \). From (E.8). Therefore, from above,

(E.10) \[ \Delta g = \frac{PVR^* - PVG^*}{\Psi} = \left( \frac{PVR^*}{PVG} - 1 \right) \pi. \]

Since \( g = \pi \) at the old steady state, we have

(E.11) \[ \frac{\Delta g}{g} = \frac{PVR^*}{PVG} - 1. \]

In terms of the constant stream of government expenditures that can be financed, we have from the government’s present value budget constraint

(E.12) \[ PVR^* = \sum_{t=0}^{\infty} \frac{G}{\prod_{s=1}^{T} (1 + \gamma_s)} \]

\[ = G \left[ \sum_{t=0}^{T} \frac{1}{\prod_{s=1}^{T} (1 + \gamma_s)} + \sum_{t=T+1}^{\infty} \frac{1}{\prod_{s=1}^{T} (1 + \gamma_s)} \right] \]

\[ = G \left[ \sum_{t=0}^{T} \frac{1}{\prod_{s=1}^{T} (1 + \gamma_s)} + \frac{1}{\prod_{s=1}^{T} (1 + \gamma_s)} \right] = G \cdot \Xi_1, \]

where \( \Xi_1 \) is a discount factor. At the old steady state, we have

(E.13) \[ PVR = \sum_{t=0}^{\infty} \frac{\bar{G}}{\prod_{s=1}^{T} (1 + \gamma_s)} = \bar{G} \sum_{t=0}^{\infty} \frac{1}{(1 + \gamma)^t} = \bar{G} \frac{1}{1 - \frac{1}{1 + \gamma}} = \bar{G} \cdot \Xi_2, \]

where \( \Xi_2 \) is a discount factor. Therefore, we can obtain \( \frac{\Delta g}{g} = \frac{G - \bar{G}}{\bar{G}} \) from (E.12) and (E.13).
E.4. Welfare Measure

Utility at the original steady state is expressed as follows:

\[ V = \sum_{t=0}^{\infty} \left( \frac{1}{1 + \rho} \right)^t U(c_t, l_t) = \sum_{t=0}^{\infty} \left( \frac{1}{1 + \rho} \right)^t \frac{1}{1 - \sigma} (c_t l_t^{\Omega})^{1-\sigma} \]

\[ = \sum_{t=0}^{\infty} \left( \frac{1}{1 + \rho} \right)^t \frac{1}{1 - \sigma} (b_t l_t^{\Omega})^{1-\sigma} \sum_{t=0}^{\infty} \left( \frac{1}{1 + \rho} \right)^t \frac{1}{1 - \sigma} (c_t l_t^{\Omega})^{1-\sigma} \]

\[ = \frac{1}{1 - \sigma} (b_t l_t^{\Omega})^{1-\sigma} H_0^{1-\sigma} \sum_{t=0}^{\infty} \frac{\{(1 - \delta_H) + \bar{m}\}^{1-\sigma}}{1 + \rho} \]

\[ = \frac{1}{1 - \sigma} (b_t l_t^{\Omega})^{1-\sigma} H_0^{1-\sigma} \frac{1}{1 - \{(1 - \delta_H) + \bar{m}\}^{1-\sigma}}. \]

On the other hand, utility after the change in the tax rate is expressed as follows:

\[ \hat{V} = \sum_{t=0}^{\infty} \left( \frac{1}{1 + \rho} \right)^t U(\hat{c}_t, \hat{l}_t) \]

\[ = \sum_{t=0}^{T} \left( \frac{1}{1 + \rho} \right)^t \frac{1}{1 - \sigma} (c_t l_t^{\Omega})^{1-\sigma} + \sum_{t=T+1}^{\infty} \left( \frac{1}{1 + \rho} \right)^t \frac{1}{1 - \sigma} (c_t l_t^{\Omega})^{1-\sigma} \]

\[ = \sum_{t=0}^{T} \left( \frac{1}{1 + \rho} \right)^t \frac{1}{1 - \sigma} (b_t H_0 \prod_{s=0}^{t-1} [(1 - \delta_H) + m_s]) l_t^{\Omega})^{1-\sigma} \]

\[ + \sum_{t=T+1}^{\infty} \left( \frac{1}{1 + \rho} \right)^t \frac{1}{1 - \sigma} [b^*(H_0 \prod_{s=0}^{T} [(1 - \delta_H) + m_s])[(1 - \delta_H) + m^*]^{1-(T+1)} l_t^{\Omega})^{1-\sigma} \]

\[ = H_0^{1-\sigma} \sum_{t=0}^{T} \left( \frac{1}{1 + \rho} \right)^t \frac{1}{1 - \sigma} (b_t l_t^{\Omega})^{1-\sigma} \prod_{s=0}^{t-1} [(1 - \delta_H) + m_s]^{1-\sigma} \]

\[ + \left( \frac{1}{1 + \rho} \right)^{T+1} \frac{1}{1 - \sigma} (b^* l_t^{\Omega})^{1-\sigma} H_0^{1-\sigma} \prod_{s=0}^{T} [(1 - \delta_H) + m_s]^{1-\sigma} \frac{1}{1 - \{(1 - \delta_H) + m^*\}^{1-\sigma}}. \]

Welfare gain or loss is measured in terms of the welfare measure of Lucas (1990) and King and Rebelo (1990): find \( \zeta \) such that

\[ V(\{(1 + \zeta) c_t, l_t\}_{t=0}^{\infty}) = V(\{c_t, l_t\}_{t=0}^{\infty}). \]

Since \( V(\{c_t, l_t\}_{t=0}^{\infty}) = V \) and \( V(\{\hat{c}_t, \hat{l}_t\}_{t=0}^{\infty}) = \hat{V} \) from above, then we have

\[ (1 + \zeta)^{1-\sigma} V = \hat{V}. \]

Therefore, the welfare measure “compensating consumption supplement” is expressed as

\[ \zeta = \left( \frac{\hat{V}}{V} \right)^{1-\sigma} - 1. \]
References


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\[ +0.0503223746076i \quad -0.0503223746076i \]

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### Table 4. Sensitivity Analysis

#### Case 1. $\sigma = 4$

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<th>Welfare Change in Consumption Term (%)</th>
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Figure 2A

Figure 2B
Figure 2C

Figure 3

Present Value of Tax Revenues evaluated by after-tax interest rate net of depreciation

Wage Tax Rate in Sector 1 (%)
Figure 4C

Welfare Change in Lucas Measure (Percent Change in Annual Consumption) vs. Wage Tax Rate in Sector 1 (%)