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A Re-examination of Cachon and Swinney (2009)

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When Is Dynamic Pricing Profitable for a Seller in the Presence of Strategic Consumers?
A Re-examination of Cachon and Swinney (2009)*

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Abstract

Pricing policy and inventory decision are essential for retailers that sell products whose demand is uncertain in advance, and whose value decreases rapidly. Studies on newsvendor model have tackled this topic over the ages. In recent years, the literature has an interest in a dynamic model where consumers strategically choose their purchase timing, and a retailer decides its prices at multiple periods. Many papers analyzing such models show that restricting markdown pricing is beneficial for a retailer in the presence of strategic consumers, while Cachon and Swinney (2009) assert that it is better for a seller to mark down optimally even in the presence of strategic consumers. In this paper, we reexamine the validity of their assertion by simplifying their model, and find that their results depend on the value of consumer surplus at the full price period which is exogenous in their paper. Thus, their claim is applicable to the case that consumers obtain large surplus from their purchase at the full price period.

Keywords: pricing policy, inventory policy, strategic consumers, newsvendor model

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1 Introduction

Inventory decision and pricing policy are essential for retailers that sell perishable products whose demand is uncertain in advance. Studies on newsvendor model have tackled this topic over the ages (See Qin et al. (2011) and Choi (2012) for reviewing the literature). In recent years, the literature has an interest in a dynamic model where consumers strategically choose their purchase timing, and a retailer decides its prices at multiple periods. Many papers dealing with such models show that committing to a fixed price path ex ante (in particular, committing to restricting markdown pricing) is beneficial for a retailer in the presence of strategic consumers, because such commitment reduces strategic consumers’ behavior to delay their purchase (See Besanko and Winston (1990); Aviv and Pazgal (2008); Dasu and Tong (2010) for example).

On the other hand, Cachon and Swinney (2009) assert that it is better for a seller to mark down optimally even in the presence of strategic consumers based on numerical analysis of their two-period model. The following is the quotation from abstract of Cachon and Swinney (2009) (CS hereafter):

We find that a retailer should generally avoid committing to a price path over the season (assuming such commitment is feasible) – committing to a markdown price (or to not mark down at all) is often too costly (inventory may remain unsold) even in the presence of strategic consumers; the better approach is to be cautious with the initial quantity and then mark down optimally.

In this paper, we reexamine the validity of their assertion by using a simplified model that allow us to examine an equilibrium of their model in detail. As a result, we find that their result crucially depends on the exogenous value of consumer surplus from purchasing a product at the full price, and restricting markdown pricing raises a retailer’s expected profit when the value of the consumer surplus is small compared to the value in the original paper. Thus, the above-mentioned CS’s claim is not general as they insist, and applicable to the case that consumers obtain large surplus from their purchase at the full price period.

The rest of this paper is organized as follows. In Section 2, we propose a simplified version of CS model, and derive a set of conditions which must be satisfied in a equilibrium. In Section 3, we conduct a numerical analysis based on the conditions derived in Sec 2, and examine the validity of the above remark by CS. Concluding remarks is in Section 4. All proofs of lemmas are in the Appendix at the end of this paper.
2 Model

In this section, first we describe the CS model and simplify the model for our detailed examination in the next two subsections. Then, we describe a perfect Bayesian equilibrium of the model by summarize the examination up to then then. Finally, we provide two benchmark cases to assess equilibrium outcomes in a numerical analysis.

2.1 Description and simplification of CS model

The CS model is a two-period model under demand uncertainty. In the model, a monopolistic retailer chooses its inventory of a product $q$ at the beginning of the first period, then sets its sale price $s$ in the second period if there exist leftovers at the beginning of the period. The unit cost to procure the product is $c$. Note that the first period price $p$ is assumed to be exogenous.\(^1\)

The CS model has three groups of consumers: Bargain-hunting, Myopic, and Strategic consumers. Let us denote by $v_B$ and $v_M$ bargain-hunting and myopic consumers’ valuations of the product, respectively. Bargain-hunting consumers intend to buy the product only at the second period as long as $v_B - s \geq 0$, while myopic consumers intend to buy the product only at the first period as long as $v_M - p \geq 0$. We denote by $v_S^i$ strategic consumers’ valuation of the product at period $i$ ($i = 1, 2$). In the CS model, it is assumed that $v_S^1 = v_M$ and $v_S^2$ is uniformly distributed in the interval $[v, \overline{v}]$. Strategic consumers choose the timing of their purchase at the beginning of the first period in order to maximize their expected surplus.

In the CS model, the sum of myopic and strategic consumers is $D$ which is a random variable. A fraction $\alpha$ of $D$ is strategic consumers. The number of bargain-hunting consumers is assumed to be unlimited. Thus, if the retailer sets the sale price such that $s \leq v_B$, then the retailer is able to clear its inventory at the end of the second period.

We simplify the CS model in order to make it more tractable. We assume that $D$ follows an uniform distribution $U[0, \overline{D}]$ where $\overline{D}$ is the upper limit of potential consumers who would buy the product at the first period. Correspondingly, we assume that the number of bargain-hunting consumers is fixed at $\overline{D}$. In addition, we only consider two cases: $\alpha = 0$ (i.e., there are myopic and bargain-hunting consumers, and no strategic consumers in the model) and $\alpha = 1$ (i.e., there are strategic and bargain-hunting consumers, and no myopic consumers in the model). CS assumed that $D$ has no upper limit.\(^2\) However, it is reasonable

\(^1\)CS extended their model to allow the retailer’s quick response (i.e., additional procurements with a higher cost before the start of 1st period are possible). However, we do not consider such an extension in this paper in order to focus on the fundamental principle that produces their result.

\(^2\)CS used gamma distributions for $D$ in their numerical analysis.
to assume that there is the upper limit of $D$, since the population in the trading area of a retailer is finite in most cases.

We formulate a simplified CS model as a dynamic game of incomplete information, and derive a perfect Bayesian equilibrium of the model. On the other hand, CS used “a subgame perfect Nash equilibrium with rational expectations” as their solution concept (see Definition 2 in CS), since they probably would like to avoid to make their analysis unnecessarily complicated.\textsuperscript{3} The reason of our choice of the equilibrium concept is that a perfect Bayesian equilibrium is standard for the game theoretic modeling to the problem we considered, and we would like to examine the simplified CS model thoroughly.

To do so, we introduce myopic and strategic consumers’ type; active and inactive. Below, we call strategic consumers when $\alpha = 1$ (myopic consumers when $\alpha = 0$) as consumers. We also call bargain-hunting consumer as bargain-hunters. To begin with, we focus on the case of $\alpha = 1$. Active consumers’ valuations of the product are the same as before (i.e., $v^1_S = v_M$ and $v^2_S \in [\underline{v}, \overline{v}]$), while those for inactive consumers are zero (i.e., $(v^1_S, v^2_S) = (0, 0)$). Consumers are uniformly distributed along the fictitious interval $[0, D]$. Each consumer knows the above facts, but she is assumed to have no knowledge about the exact location on the interval where she lie.

Nature decides the type of each consumer as follows. It determines the number of active consumers $D_A$ based on a sample drawn from the uniform distribution $D_A \sim U [0, D]$ whose density function is

$$f(D_A) = \frac{1}{D}. \quad (1)$$

After that, consumers located in $[0, D_A]$ know that they are active, while other consumers know that they are inactive. Nature attaches each consumer their reservation price of the product. If a consumer is active, then it holds that $v^1_S = v_M$ and $v^2_S$ is set to a value drawn from the interval $[\underline{v}, \overline{v}]$ such that $v^2_S$ is uniformly distributed among $[\underline{v}, \overline{v}]$ as a whole. If a consumer is inactive, then it holds that $(v^1_S, v^2_S) = (0, 0)$.

Since each consumer does not know her location on the interval, she is not possible to know other consumers’ type, and the realized value of $D_A$ directly from her type. However, if a consumer is a Bayesian decision maker, she can infer the distribution of $D_A$ given her type using Bayes theorem as the next lemma shows.

Lemma 1 The prior probability that a consumer is active is $\frac{1}{2}$. If a consumer is active, then the consumer updates the density function of the number of active consumers from

\textsuperscript{3}See footnote 11 in CS.
\[ f(D_A) = \frac{1}{D} \text{ to} \]
\[ f(D_A|\text{active}) = \frac{2D_A}{(D)^2}, \quad 0 \leq D_A \leq 1. \]  
(2)

From this lemma, we find that for each consumer the prior probability that she is active is \(\frac{1}{2}\). This seems reasonable. We also find that if a consumer becomes active, then she infers that it is more probable that the number of active consumers is more than 0.5\(D\). This may seem extraordinary since only one consumer’s information that she is active changes the posterior distribution of \(D\) drastically. However, this extremeness comes from the fact that the prior distribution of \(D_A\) is assumed to be uniform in order to make the model tractable.

After each consumer’s type is determined, the game proceeds to the beginning of the first period. Before that, we assume that

**Assumption 1**

\[ 0 < v_B < c < \underline{v} < p < v_M. \]  
(3)

This assumption means the following. First, an active consumer’s surplus \(v_M - p\) is positive if she can purchase the product at the full price period. Note that there are the cases that part of active consumers can not purchase the product even if they intend to do, since the rationing may occur at 1st period. We explain it below soon. Second, the upper limit of the value of the product at the second period is lower than the full price (\(\overline{v} < p\)). Thus, the retailer has to set a price lower than the full price in order to make sales at the second period. In addition, the lower limit of the value of the product at the second period is higher than the procurement cost (\(\underline{v} > c\)). Thus, the retailer does not get a negative margin when it sells the product at the price equal to or higher than \(\underline{v}\), although there may be leftover at the end of the second period. Finally, in order to sell the product to bargain-hunters, the retailer has to set its sale price equal to or lower than \(v_B\), which results in the situation that the retailer’s second period sales are positive but smaller than the cost it incurs at the beginning of the first period since \(v_B < c\).

We briefly explain here how the game proceeds after each consumer’s type determined. At the start of the first period, the retailer and consumers make their decision simultaneously given that the value of \(D_A\) is unknown. The retailer chooses its inventory of the product \(q\) whose unit procurement cost is \(c\). Active consumers choose their purchase timing; to buy at 1st period or at 2nd period. As we will show later, given \(q\) there exists some \(\hat{v} \in [\underline{v}, \overline{v}]\) such that an active consumer with \(v_S^2 = \hat{v}\) is indifferent between 1st period purchase and 2nd period purchase. Active consumers with \(v_S^2 < \hat{v}\) will do their shopping at 1st period, while those with \(v_S^2 \geq \hat{v}\) will do their shopping at 2nd period.\(^4\) Inactive consumers who have no

\(^4\)We assume that consumers with \(v_S^2 = \hat{v}\) will do their shopping at 2nd period. As CS noted, indifferent
value for the product decide not to buy the product at either period, since it is their dominant strategy. Thus, the 1st period demand for the product is $D_1 = (\hat{v} - \underline{v}) D_A$. If $q \leq D_1$, the game stops at the end of 1st period. We assume that each consumer who intends to purchase the product at the 1st period is able to do so with the probability, $\min\left(\frac{q}{D_1}, 1\right)$. If $q < D_1$, then only part of them can actually purchase it. If $q > D_1$, all consumers who intend to buy the product at 1st period can actually do it. There remains an inventory of the product at the end of this period which we denote as $I$. Then, the game moves on to 2nd period. At the beginning of the second period, the realized values of $q$ and $\hat{v}$ become public information.\(^5\) In addition, $I$ is assumed to be observable to all players. Thus, the value of $D_A = D$ can also be inferred by the relation, $I = q - (\hat{v} - \underline{v}) D$. Then, the retailer sets the sale price $s$ given $D$. After observing this price, consumers with $v^*_S \geq b\hat{v}$ and bargain-hunters decide to purchase the product or not.

Next, we set up the model in detail and derive its equilibrium. We analyze the model backward, thus start our examination from the second period.

2.2 Second Period

Period 2 opens when $I > 0$ given ($\hat{v}, q$). The retailer and active consumers are able to know the realized value of $D_A = D \in [0, \overline{D}]$ by the relation, $I = q - (\hat{v} - \underline{v}) D$. Thus, the belief about $D_A$ is reduced to

$$\Pr(D_A = D | q, \hat{v}, I) = 1 \quad \text{for some } D \in [0, \overline{D}] .$$

With this belief, the retailer sets the sale price $s$.

Consumers with $v^*_S \geq \hat{v}$ and bargain-hunters intend to purchase the product if their valuations of the product are higher than or equal to $s$. Thus, the 2nd period demand is described as

$$D_2 = \begin{cases} 
(\bar{v} - s) D & \text{if } \hat{v} \leq s \leq \bar{v}, \\
(\bar{v} - \hat{v}) D & \text{if } v_B < s < \hat{v}, \\
(\bar{v} - \hat{v}) D + \overline{D} & \text{if } s \leq v_B .
\end{cases}$$

\(^5\)It seems reasonable that $q$ becomes public, while it may be debatable that $\hat{v}$ becomes public in reality. However, we can consider that all players (i.e., the retailer and (strategic) consumers) are rational enough to solve the model and obtain ($\hat{v}, q$) in equilibrium. With this thinking, the second period pricing problem becomes a simple static one without uncertainty. Without it, the retailer must infer the joint posterior probability of ($\hat{v}, D$) before setting to its sale price.
The retailer’s second period revenue is given as

\[ R(s, I) = s \min (D_2, I). \]  \hspace{1cm} (6)

Since the procurement cost of the product has already been sunk, the retailer maximizes \( R(s, I) \) with respect to \( s \).

To simplify the retailer’s maximization problem, we assume that

**Assumption 2**

\[ \bar{v} - \underline{v} = 1, \]  \hspace{1cm} (7)

\[ \frac{\bar{v}}{2} \leq \underline{v}. \]  \hspace{1cm} (8)

Without changing the result obtained by CS qualitatively, this assumption simplifies CS’s Lemma 2 as follows:

**Lemma 2** Suppose Assumptions 1 – 2. Then, the optimal sale price to maximize (6) is presented as

\[ s^* (D) = \begin{cases} 
  s_h = (\frac{D - q}{D}) + \hat{v} & \text{if } D_m (q) < D < D_h (q, \hat{v}), \\
  s_m = \hat{v} & \text{if } D_l (q, \hat{v}) < D \leq D_m (q), \\
  s_l = v_B & \text{if } D \leq D_l (q, \hat{v}),
\end{cases} \]  \hspace{1cm} (9)

where, given \((q, \hat{v})\), the threshold demand levels, \( D_h \), \( D_m \) and \( D_l \) that change the optimal sale price are defined as

\[ D_h (q, \hat{v}) = \min \left( \frac{q}{\hat{v} - \underline{v}}, \frac{\hat{v}}{D} \right), \quad D_m (q) = q, \quad D_l (q, \hat{v}) = \frac{v_B q}{v_B + (\hat{v} - \underline{v}) (\hat{v} - v_B)} \]  \hspace{1cm} (10)

respectively.

From this lemma, we find the following. First, if the realized demand \( D \) (the number of active consumers) is large enough to satisfy that \( D_m (q, \hat{v}) < D \leq D_h (q, \hat{v}) \), then the retailer sets the high sale price ( \( s_h = (\frac{D - q}{D}) + \hat{v} \) ) in order to clear its inventory \( I \) by selling to part of remaining consumers whose valuation of the product is high. Second, if it is the intermediate level such that \( D_l (q, \hat{v}) < D \leq D_m (q, \hat{v}) \), then the retailer sets the in-between sale price ( \( s_m = \hat{v} \) ) and sells the product to remaining consumers. In this case, there exist leftover products at the end of the period. Finally, if it is so small that \( D \leq D_l (q, \hat{v}) \), then the retailer sets the low sale price ( \( s_l = v_B \) ) in order to let bargain-hunters purchase the product. As a result, the retailer sells the product to both remaining consumers and part of bargain-hunters, and clears its inventory.
There are two differences between the above Lemma and CS’s Lemma 2 other than simplified expressions of equations. First, the highest threshold, \( D_h(q, \tilde{v}) \) in (10) has \( \bar{D} \) (the upper limit of potential demand) as its element, while the counterpart in CS’s Lemma 2 does not. This is because we use the probability distribution with the finite upper limit, while CS use the probability distribution without it. In our model, period 2 opens when \( I > 0 \) (\( \leftrightarrow D < \frac{q}{v-q} \)). The inequality, \( \frac{q}{v-q} > \bar{D} \), holds if \( \tilde{v} \) is very low (i.e., many active consumers intend to do shopping at 2nd period). This is taken into account in the expression of \( D_h(q, \tilde{v}) \) in (10).

Second, the intermediate optimal sale price is \( s_m = \tilde{v} \), while the counterpart in CS is

\[
s_m = \arg \max_{s \geq \tilde{v}} (\bar{v} - s) = \begin{cases} \bar{v} & \text{if } \bar{v} > \tilde{v}, \\ \tilde{v} & \text{if } \frac{\bar{v}}{2} \leq \tilde{v}, \end{cases}
\]

which depends on the values of \( \bar{v} \) and \( \tilde{v} \). The fact that \( s_m \) changes depending on the relative size of \( \bar{v} \) and \( \tilde{v} \), and the latter is determined in equilibrium complicates the retailer’s inventory choice in 1st period. (8) is a sufficient condition for the intermediate optimal sale price to be always \( s_m = \tilde{v} \). Actually, CS use the following as the parameter values of their numerical analysis: 6

\[
[v, \bar{v}] \in \{ [2, 10], [3, 4], [6, 7], [9, 10] \}.
\]

The three cases other than \([2, 10]\) satisfy (7) and (8). Thus, Assumption 2 will not alter the result which we will derive later significantly compared to CS’s result.

### 2.3 First Period

At the beginning of the first period, the retailer and consumers make their decision simultaneously given that the value of \( D_A \) is unknown. For their decision making, they take into account the second period strategies with the belief (4) that active consumers, bargain-hunters and the retailer adopt, which are summarized as (5) and (9) respectively.

#### 2.3.1 Consumers’ strategy

First, we examine consumers’ strategy given the first period inventory \( q \). Inactive consumers do not go shopping at either period, since their valuation of the product at each period is zero. This is the dominant strategy for each inactive consumer irrespective of other players’ strategies and the value of \( D_A \). On the other hand, active consumers have positive valuation of the product, and choose their purchase timing.

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6See Table 2 in their paper.
Suppose that there exists some $\hat{v} \in [\underline{v}, \overline{v}]$ such that an active consumer with $v_S^2 = \hat{v}$ is indifferent between 1st period purchase and 2nd period purchase and that $\hat{v}$ segments active consumers into two groups such that those with $v_S^2 < \hat{v}$ will do their shopping at 1st period, while those with $v_S^2 \geq \hat{v}$ will do their shopping at 2nd period. We will show later that such a threshold exists under certain conditions.

Then, when $D_h (q, \hat{v}) < \overline{D}$ ( $\iff \hat{v} > \underline{v} + \frac{q}{\overline{D}}$ ), the indifferent active consumer’s expected surplus of her 1st period purchase is given as

$$EU_1 (\hat{v}; q) = \int_0^{\overline{D}} \min \left(1, \frac{q}{D_A (\hat{v} - \underline{v})}\right) (v_M - p) f (D_A |active) dD_A$$

$$= \int_0^{\overline{D}} \min \left(1, \frac{q}{D_A (\hat{v} - \underline{v})}\right) (v_M - p) \frac{2D_A}{(\overline{D})^2} dD_A$$

$$= \int_0^{D_h (q, \hat{v})} (v_M - p) \frac{2D_A}{(\overline{D})^2} dD_A + \int_{D_h (q, \hat{v})}^{\overline{D}} \frac{q}{(\hat{v} - \underline{v})} (v_M - p) \frac{2}{(\overline{D})^2} dD_A$$

$$= (v_M - p) \left( \frac{D_h (q, \hat{v})}{\overline{D}} \right) \left( 2 - \frac{D_h (q, \hat{v})}{D} \right), \quad (11)$$

which is less than $v_M - p$, since the probability that she cannot purchase the product is positive. On the other hand, when $D_h (q, \hat{v}) \geq \overline{D}$ ( $\iff \hat{v} \leq \underline{v} + \frac{q}{\overline{D}}$ ), the surplus is given as

$$EU_1 (\hat{v}; q) = \int_0^{\overline{D}} (v_M - p) \frac{2D_A}{(\overline{D})^2} dD_A$$

$$= \frac{v_M - p}. \quad (12)$$

Summing up (11) and (12), we obtain

$$EU_1 (\hat{v}; q) = (v_M - p) \min \left( \frac{q}{(\hat{v} - \underline{v}) \overline{D}}, 1 \right) \left[ 2 - \min \left( \frac{q}{(\hat{v} - \underline{v}) \overline{D}}, 1 \right) \right]. \quad (13)$$

For deriving the indifferent active consumer’s expected surplus of 2nd period, we assume that, when the retailer sets its sale price equal to $v_B$ at the second period, the retailer sells the product to active consumers ahead of bargain-hunters until it clears $I$. In other words, in the queue consisted of active consumers with $v_S^2 \geq \hat{v}$ and bargain-hunters, the former is at its front, and the retailer serves from the front of the queue until $I$ disappears. On the other hand, CS considered a more general situation where active consumers are uniformly distributed among the first part of $\frac{(\hat{v} - \underline{v}) \overline{D}}{\theta}$ in the line where $\theta (\in [0, 1])$ is the parameter that stands for the level of optimism of consumers. When $\theta < 1$, part of consumers cannot purchase the product when $s = v_B$. The larger the value of $\theta$ is, the more optimistic consumers become.
When $\theta = 1$, all consumers can purchase the product at the bargain price. CS derived the probability that the indifferent active consumer can purchase the product in Lemma 4 in their paper. Note that this probability changes depending on the value of $\theta$ whose threshold $\theta_c$ is defined in Lemma 4 in CS as

$$\theta_c = \frac{s_l}{s_m} = \left\{ \begin{array}{ll} \frac{2v_B}{\bar{v}} & \text{if } \frac{\bar{v}}{2} > \hat{v}, \\ \frac{v_B}{\bar{v}} & \text{if } \frac{\bar{v}}{2} \leq \hat{v}. \end{array} \right.$$ 

In CS’s setting, $\theta_c$ is constant when $\frac{\bar{v}}{2} > \hat{v}$. Otherwise it changes depending on $\hat{v}$. The fact that $\theta_c$ changes depending on the relative value of $\bar{v}$ and $\hat{v}$, and the latter is determined in equilibrium complicates active consumers’ choice at 1st period. Thus, in order to simplify our analysis, we stick to the assumption that the retailer sells the product to active consumers first which means $\theta = 1$ and active consumers are the most optimistic.

Then, from Lemma 2, we find that a chance for the indifferent active consumer to purchase the product at the price $v_B$ and obtain a positive surplus comes with the probability

$$\Pr (D_A \leq D_l (q, \hat{v}) | q, \hat{v}, I) = \int_0^{D_l (q, \hat{v})} f (D_A | \text{active}) dD_A$$

$$= \frac{1}{(D)^2} \left[ \frac{v_B q}{v_B + (\bar{v} - \hat{v}) (\hat{v} - v_B)} \right]^2. \quad (14)$$

Thus, we obtain her expected surplus of 2nd period purchase as

$$EU_2 (\hat{v}; q) = (\hat{v} - v_B) \Pr (D_A \leq D_l (q, \hat{v}) | q, \hat{v}, I)$$

$$= \frac{\hat{v} - v_B}{(D)^2} \left[ \frac{v_B q}{v_B + (\bar{v} - \hat{v}) (\hat{v} - v_B)} \right]^2. \quad (15)$$

Before deriving the threshold $\hat{v}$ that segments consumers given $q$, we make the next assumption:

**Assumption 3**

$$v_M - p < \bar{v} - v_B, \quad (16)$$

$$v_M - p > \bar{v} - v_B, \quad (17)$$

$$\bar{v} - v_B > \frac{1}{2}. \quad (18)$$

(16) means that the active consumer whose second period valuation of the product is the highest chooses 2nd period purchase if she can buy the product with certainty, while
(17) means that the active consumer whose second period valuation of the product is the lowest chooses 1st period purchase, if she can purchase it with certainty. These conditions are reasonable. (18) is a sufficient condition for \( \hat{v} \) to be unique.\(^7\)

Then, we obtain the following lemma:

**Lemma 3** Suppose Assumptions 1 – 3. Then, given \( q \), (13) is a nonincreasing function of \( \hat{v} \) while (15) is an increasing function of \( \hat{v} \). As a result, there exists some (the unique) \( \hat{v}^* \in [\underline{v}, \overline{v}] \) such that

\[
EU_1(\hat{v}^*; q) = EU_2(\hat{v}^*; q)
\]

if q satisfies

\[
\frac{2\overline{D}}{1 + \left(\frac{\overline{v}-v_B}{v_M-p}\right)} < q < \overline{D} \left(\frac{v}{v_B}\right) \sqrt{\frac{v_M-p}{\overline{v}-v_B}};
\]

(19)

if \( q \) is small enough to satisfy

\[
q \leq \frac{2\overline{D}}{1 + \left(\frac{\overline{v}-v_B}{v_M-p}\right)},
\]

(20)

then \( EU_2(\overline{v}; q) \leq EU_1(\overline{v}; q) \) and \( \hat{v}^* = \overline{v} \); if \( q \) is large enough to satisfy

\[
q \geq \overline{D} \left(\frac{v}{v_B}\right) \sqrt{\frac{v_M-p}{\overline{v}-v_B}};
\]

(21)

then \( EU_1(\underline{v}; q) \leq EU_2(\underline{v}; q) \) and \( \hat{v}^* = \underline{v} \).

In the following, given the threshold \( \hat{v}^* \) determined in Lemma 3, we examine that there exists for an active consumer to deviate from a situation where consumers with \( v_S^2 < \hat{v} \) will do their shopping at 1st period, while those with \( v_S^2 \geq \hat{v} \) will do their shopping at 2nd period. Note that the retailer has no incentive to procure the product more than \( \overline{D} \) (i.e., the upper limit of the number of active consumers). We rule out the case of (21) below.

Suppose that (19) holds. In addition, suppose that all consumers except a chosen consumer do their shopping at 1st period (2nd period) if their 2nd period valuation of the product satisfies \( v_S^2 < \hat{v} \) (\( v_S^2 \geq \hat{v} \)). Since the the chosen consumer’s measure is zero, other consumers and the retailer are not affected by her action. Thus, the second period actions, (5) and (9) remain the same if she changes her purchase timing.

We denote by \( EU_i(\overline{v}_S^2, \hat{v}; q) \) the expected surplus from a purchase at period \( i \) by the chosen consumer whose product valuation at 2nd period is \( \overline{v}_S^2 \). As for her expected surplus at 1st period, it holds

\[
EU_1(\overline{v}_S^2, \hat{v}; q) = EU_1(\hat{v}; q) \quad \text{for any } \overline{v}_S^2 \in [\underline{v}, \overline{v}] .
\]

\(^7\)if (18) does not hold, then there may exist multiple \( \hat{v} = \hat{v}^* \in [\underline{v}, \overline{v}] \) satisfying \( EU_1(\hat{v}; q) = EU_2(\hat{v}; q) \).
That is, the chosen consumer with an arbitrary value of $v_2^S$ obtains the expected surplus of a consumer with $v_2^S = \hat{v}^*$ if she intends to purchase the product at 1st period, since $v_M - p$ and the purchase probability $\min\left(\frac{q}{v^* - v_ID}, 1\right)$ are independent of $v_2^S$ (Note that the purchase probability is determined by the action of the other consumers as a whole). Next, as for her expected surplus at 2nd period, it holds

$$EU_2(\bar{v}_S^2, \hat{v}^*; q) > EU_2(v_2^*; q) \iff \bar{v}_S^2 > \hat{v}^*.$$ 

That is, the chosen consumer’s 2nd period expected surplus is higher than the expected surplus of the indifferent consumer with $v_2^S = v_2^*$ if $\bar{v}_S^2 > \hat{v}^*$. This is because she has a chance to obtain a positive surplus even when the retailer’s sale price is not equal to $v_B$. By contrast, if $\bar{v}_S^2 < \hat{v}^*$, then the chosen consumer’s 2nd period expected surplus is lower than the expected surplus of the indifferent consumer with $v_2^S = v_2^*$. This is because $\bar{v}_S^2 - v_B < \hat{v}^* - v_B$ and the probability (14) does not depend on her behavior.

Next, suppose that (20) holds. In addition, suppose that all consumers except a consumer do their shopping at 1st period. In this case, it holds

$$EU_1(\bar{v}_S^2, \bar{v}; q) = EU_1(v; q) \quad \text{for any } \bar{v}_S^2 \in [\bar{v}, \bar{v}],$$

and

$$EU_2(\bar{v}_S^2, \bar{v}; q) < EU_2(v; q) \quad \text{for any } \bar{v}_S^2 \in [\bar{v}, \bar{v}].$$

Thus, the chosen consumer with an arbitrary value of $\bar{v}_S^2$ intends to purchase the product at 1st period.

Therefore, we summarize the above argument as follows

**Proposition 1** Suppose Assumptions 1 – 3. When (19) holds, given the threshold $\hat{v}^*$ determined by $EU_1(\hat{v}^*; q) = EU_2(\hat{v}^*; q)$, consumers with $v_2^S < \hat{v}$ do their shopping at 1st period, while those with $v_2^S \geq \hat{v}$ will do their shopping at 2nd period. When (20) holds, all consumers do their shopping at 1st period. When (21) holds, all consumers do their shopping at 2nd period, though the retailer does not have an incentive to hold such a large inventory.

Lemma 3 and Proposition 1 correspond to CS’s Lemma 1. In our analysis, we take into account the case when the rationing occurs at 1st period, while it seems that CS did not in their analysis. Actually, in CS’s setting, the rationing will occur at 1st period since CS assume that $D$ has no finite upper limit and thus there is some $D_1$ such that $D_1 > q$ for any positive $q$. We also derive conditions about the retailer’s inventory that determines consumers’ behavior: (19)–(21), which is useful to conduct a numerical analysis of the model.
2.3.2 Retailer’s strategy

Next, we describe the retailer’s expected profit at the start of period 1 given \( \hat{v} \). Note that the retailer has entirely no knowledge about the number of active consumers at the time. Thus, it uses the density of uniform distribution \( f(D_A) = \frac{1}{D} \) to evaluate its expected profit.

Taken into account the second period actions, (5) and (9), it is given as follows:

\[
\pi(q; \hat{v}) = p \int_0^{D_h(q, \hat{v})} (\hat{v} - v) \frac{DA}{D} dD_A + p \int_{D_h(q, \hat{v})}^D \left( \frac{q}{D} \right) dD_A - cq
\]

\[
+ v_B \int_0^{D_l(q, \hat{v})} [q - (\hat{v} - v) D_A] \left( \frac{1}{D} \right) dD_A + \hat{v} \int_{D_l(q, \hat{v})}^{D_m(q)} (\hat{v} - \bar{v}) \frac{DA}{D} dD_A
\]

\[
+ \int_{D_m(q)}^{D_h(q, \hat{v})} \left( 1 - \frac{q}{D_A} + \hat{v} \right) [q - (\hat{v} - v) D_A] \left( \frac{1}{D} \right) dD_A,
\]

which is reduced to

\[
\pi(q; \hat{v}) = p \frac{(\hat{v} - v)}{D} \int_0^{D_h(q, \hat{v})} D_A dD_A + \frac{pq}{D} \int_{D_h(q, \hat{v})}^D dD_A - cq
\]

\[
+ \frac{v_B}{D} \int_0^{D_l(q, \hat{v})} [q - (\hat{v} - v) D_A] dD_A + \frac{\hat{v} (\bar{v} - \hat{v})}{D} \int_{D_l(q, \hat{v})}^{D_m(q)} D_A dD
\]

\[
+ \frac{1}{D} \int_{D_m(q)}^{D_h(q, \hat{v})} \left( 1 - \frac{q}{D_A} + \hat{v} \right) [q - (\hat{v} - v) D_A] dD_A.
\]

Differentiating (22) with respect to \( q \) makes

\[
\frac{d\pi(q; \hat{v})}{dq} = p \left( 1 - \frac{\frac{q}{\hat{v} - \bar{v}}}{D} \right) - c + v_B \frac{D_l(q, \hat{v})}{D} + \frac{1}{D} \int_{D_m(q)}^{D_h(q, \hat{v})} \left[ 2 \left( 1 - \frac{q}{D_A} + \hat{v} \right) - \bar{v} \right] dD_A,
\]

(23)

when \( \frac{q}{\hat{v} - \bar{v}} < \bar{D} \), and

\[
\frac{d\pi(q; \hat{v})}{dq} = -c + v_B \frac{D_l(q, \hat{v})}{D} + \frac{1}{D} \int_{D_m(q)}^{D_h(q, \hat{v})} \left[ 2 \left( 1 - \frac{q}{D_A} + \hat{v} \right) - \bar{v} \right] dD_A,
\]

(24)

when \( \frac{q}{\hat{v} - \bar{v}} \geq \bar{D} \). Using the definition of \( D_h(q, \hat{v}) \) in (10), (23) and (24), we can summarize the first-order condition for the retailer’s optimal order quantity as follows

\[
\frac{d\pi(q; \hat{v})}{dq} = p \left( 1 - \frac{D_h(q, \hat{v})}{D} \right) - c + v_B \frac{D_l(q, \hat{v})}{D} + \frac{1}{D} \int_{D_m(q)}^{D_h(q, \hat{v})} \left[ 2 \left( 1 - \frac{q}{D_A} + \hat{v} \right) - \bar{v} \right] dD_A = 0.
\]

(25) is essentially the same as the corresponding equation in CS’s Lemma 3. CS proved
that the retailer’s optimal order quantity \( q^* (\hat{v}) \) is determined by the first-order condition given \( \hat{v} \), and it is unique. Instead, in our numerical analysis below, we derive \( q^* (\hat{v}) \) by two approaches; one is directly from (22) and the other is by using (25). After that, we examine its uniqueness.

2.4 Equilibrium

By summing up the above argument, we can describe a perfect Bayesian equilibrium of the model as follows. The pair \((q^*, \hat{v}^*)\) is determined by

\[
q^* = \arg \max \pi (q; \hat{v}^*), \tag{26}
\]
\[
\hat{v}^* = \arg \max (EU_1 (\hat{v}; q^*), EU_2 (\hat{v}; q^*)). \tag{27}
\]

**Period 1:** With the belief (2) and given \( q^* \), an active consumer whose second period valuation of the product is lower (higher or equal to) than \( \hat{v}^* \) intends to purchase the product at this period (postpones her choice to the next period). With the belief (1) and given \( \hat{v}^* \), the retailer chooses its inventory \( q^* \) maximizing its profit (22). When \( q^* > (\hat{v}^* - v) D \), the game proceeds to period 2.

**Period 2:** With the belief (4) and given \((q^*, \hat{v}^*)\), the retailer sets its sale price by (9). With the belief (4) the active consumers with \( v_2^S \geq \hat{v}^* \) and bargain-hunters intend to purchase the product if their valuation of the product is higher or equal to the sale price determined by (9).

Inactive consumers do not plan to purchase the product at either period.

2.5 Two benchmark cases

Before conducting a numerical analysis, we provide two benchmark cases to assess equilibrium outcomes in a numerical analysis of the next section. First, we consider the case that consumers are strategic (i.e., \( \alpha = 1 \)) and the retailer adopts a static pricing policy (i.e., it commits not to mark down its price \( p \) even if there remain unsold products at the end of 1st period). Given such a pricing policy, all active strategic consumers’ dominant strategy is to buy the product at 1st period (i.e., \( \hat{v} = \overline{v} \)) since \( v_M > \overline{v} \) from (3). Thus, the 1st period demand for the product is \( D_A = D \). If \( q \leq D \), we assume that each active consumer is able to purchase the product with the probability \( \min \left( \frac{q}{D}, 1 \right) \). In this case, the retailer’s expected
profit at the start of 1st period is given as

\[
\hat{\pi}(q) = p \int_0^q \frac{D_A}{D} dD + p \int_q^\infty \frac{\bar{D}}{D} dD - cq
\]

\[
= pq - \frac{pq^2}{2D} - cq.
\]  

(28)

Note that there is no expected revenue in 2nd period even when \( q > D \), since the retailer does not mark down its price in order to sell the product to bargain-hunters. Maximizing (28) with respect to \( q \) yields

\[
\frac{d\hat{\pi}(q)}{dq} = p - c - \frac{p}{D} q = 0,
\]

\[
q^{**} = \frac{p - c}{p} D.
\]  

(29)

Substituting (29) into (28), we obtain

\[
\hat{\pi}(q^{**}) = \frac{(p - c)^2}{2p} \bar{D}.
\]  

(30)

Next, we consider the case that consumers are myopic (i.e., \( \alpha = 0 \)) and the retailer adopts a dynamic pricing policy. Myopic consumers have no value for the product at 2nd period. If they are active, buying the product at the first period is their dominant strategy. In this case, the 1st period demand for the product is \( D_A = D \). If \( q \leq D \), the game stops at the end of 1st period. We assume that each active consumer is able to purchase the product with the probability \( \min\left(\frac{q}{D}, 1\right) \). If \( q < D \), then only part of them can actually purchase it. If \( q > D \), the game proceeds to 2nd period.

In the second period, the retailer sells remaining inventory only to bargain-hunters. To do so, it sets the sale price equal to the bargain-hunters’ value for the product \( (s = v_B) \). Then the inventory clears, which is better for the retailer than the inventory remains at the end of the second period.

Taking the above into account, the retailer’s expected profit at the start of 1st period is
\[ \pi(q) = p \int_0^q \frac{D_A}{D} dD + pq \int_q^D \frac{1}{D} dD - cq + v_B \int_0^q (q - D_A) \frac{1}{D} dD \]

\[ = \frac{pq^2}{2D} + \frac{pq}{D} (D - q) - cq + \frac{v_B q^2}{2D} \]

\[ = -\frac{(p - v_B) q^2}{2D} + (p - c) q. \] (31)

Note that there exists the expected 2nd period revenue, \( v_B \int_0^q (q - D) \frac{1}{D} dD \) in (31) while there does not in (30). Maximizing (31) with respect to \( q \) yields

\[ \frac{d\pi(q)}{dq} = -\frac{(p - v_B) q}{D} + (p - c) = 0, \]

\[ q^* = \frac{p - c}{p - v_B} D. \] (32)

We find that \( q^* < D \) since \( v_B < c \). If \( v_B = c \), then it holds that \( q^* = D \). That is, the retailer holds its inventory equal to the upper limit of possible demand at the start of 1st period because, if unsold inventories occur at the end of this period, it can sell them at its procurement cost at the next period. Substituting (32) into (31), we obtain

\[ \pi(q^*) = \frac{(p - c)^2}{2(p - v_B)} D. \] (33)

From (29) and (32), we obviously find that \( q^{**} < q^* \). In addition, from (30) and (33), we find that \( \tilde{\pi}(q^{**}) < \pi(q^*) \). When the retailer commits not to mark down its price and stick to \( p \), active strategic consumers’ action become the same as that of myopic consumers. That is, all active strategic consumers purchase the product at 1st period (i.e., \( \tilde{v} = \overline{v} \)). However, since the retailer cannot clear its inventory at 2nd period, it reduces 1st period inventory compared to the case that it faces myopic consumers. As a result, the retailer’s expected profit in the first case becomes smaller than that in the second case.

3 Numerical Analysis

We conduct a numerical analysis based on the conditions derived in Sec 2, and examine the validity of the remark by CS mentioned in Sec 1.\(^8\) In the analysis, we calculate the following:

(1) the retailer’s optimal inventory \( q^* (\tilde{v}) \) given \( \tilde{v} \) that is an element of the arithmetic sequence

\(^8\)The authors are preparing to make public the code to replicate the results in this section.
from $v$ to $\overline{v}$ whose common difference of successive numbers is 0.001; (2) the indifference active consumer’s reservation price at 2nd period $\tilde{v}^*$ ($q$) given $q$ that is an element of the arithmetic sequence from 0 to $\overline{D}$ whose common difference of successive numbers is 0.001$\overline{D}$; (3) The pair $(q^*, \tilde{v}^*)$ determined by (26) and (27); (4) the area in $[0, \overline{D}] \times [v, \overline{v}]$ that yields the expected profit that the retailer obtains by adopting the dynamic pricing is larger than that in the first benchmark case where the retailer adopts the static pricing policy. Throughout the analysis, in order to focus on how the shape of $q^*$ ($\overline{v}$) and $\tilde{v}^*$ ($q$) change when $v_M$ varies, the following parameters are fixed:

$$\overline{D} = 1, \quad p = 10, \quad c = 2, \quad v_B = 1, \quad v = 8, \quad \overline{v} = 9.$$ 

First, setting $v_M = 15$, we obtain Figure 1 and $(q^*, \tilde{v}^*) = (??, ??)$. This figure shows that almost active consumers intend to purchase the product at the first period. In fact, when $q$ is less than about 6.76 from (20), all active consumers intend to purchase the product at the this period. In the green area of this figure, the expected profit that the retailer obtains by adopting the dynamic pricing is larger than that in the first benchmark case where the retailer adopts the static pricing policy. Note that $q^*$ ($\overline{v}$) corresponds to (32) in the second benchmark case because all active consumers intend to purchase the product at the first period and the retailer can sell all leftover to bargain-hunters. We find that the dynamic price is profitable for the retailer since $(q^*, \tilde{v}^*)$ is inside of the green area.

Next, setting $v_M = 12$, we obtain Figure 2 and $(q^*, \tilde{v}^*) = (8.42, 8.91)$. This figure shows that the number of active consumers that intend to purchase the product at the first period decreases compared to the previous case since the expected surplus at the first period decreases compared to the previous case. We find that the dynamic price is still profitable for the retailer since $(q^*, \tilde{v}^*)$ is inside of the green area.
Similarly, setting $v_M = 11$, we obtain Figure 3 and $(q^*, \hat{v}^*) = (8.17, 8.84)$. This figure shows that the number of active consumers that intend to purchase the product at the first period decreases compared to the previous two cases. We find that the dynamic price is still profitable for the retailer since $(q^*, \hat{v}^*)$ is inside of the green area.

Finally, setting $v_M = 10.5$, we obtain Figure 4 and $(q^*, \hat{v}^*) = (0.798, 8.72)$. We find that the dynamic price is not profitable for the retailer since $(q^*, \hat{v}^*)$ is inside of the green area. In this case, many active consumers delay to purchase the product until the retailer lowers its price since the expected surplus at the first period is very small ( $v_M - p = 0.5$). Thus, for the retailer the static pricing is profitable compared to the dynamic pricing.

As the above analysis suggests, CS’s claim that the dynamic pricing is profitable than the static pricing depends on the expected consumer surplus of the first period ( $v_M - p$) which is exogenous in the original paper. As long as $v_M - p$ is fixed at a large value, CS’s claim tends to be guaranteed even if the values of other parameters are changed. From Figures ??–??, we also find that the green area where the dynamic pricing is profitable for the retailer than the static pricing is relatively small in $[0, \overline{D}] \times [\underline{v}, \overline{v}]$. Therefore, we consider that CS’s claim is not general as they insist.

4 Concluding Remarks

Pricing policy and inventory decision are essential for retailers that sell products whose demand is uncertain in advance, and whose value decreases rapidly. Studies on newsvendor model have tackled this topic over the ages. In recent years, the literature has an interest in a dynamic model where consumers strategically choose their purchase timing, and a retailer decides its prices at multiple periods. Many papers analyzing such models show that restricting markdown pricing is beneficial for a retailer in the presence of strategic consumers, while
Cachon and Swinney (2009) assert that it is better for a seller to mark down optimally even in the presence of strategic consumers. In this paper, we reexamined the validity of their assertion by simplifying their model, and found that their results depend on the value of consumer surplus at the full price period which is exogenous in their paper and that their claim is applicable to the case that consumers obtain large surplus at the full price period. Thus, Cachon and Swinney (2009)’s claim is not general as they insist, and should be cautiously utilized.
Appendix

Proof of Lemma 1

The probability that a consumer is active conditional that $D_A = D$ is given as

$$
\Pr(\text{active} \mid D_A = D) = \int_0^D f(D_A) \, dD_A = \frac{D}{D}.
$$

Thus, the probability that she is active prior to the realization of $D$ is derived as

$$
\Pr(\text{active}) = \int_0^D \Pr(\text{active} \mid D_A) f(D_A) \, dD_A = \int_0^D \left( \frac{D_A}{D} \right) \left( \frac{1}{D} \right) \, dD_A
$$

$$
= \frac{1}{(D)^2} \left[ \frac{D_A^2}{2} \right]_0^D = \frac{1}{2}.
$$

Therefore, if a consumer is active, she can infer based on Bayes theorem that the posterior density function of the number of active consumers is represented as

$$
f(D_A \mid \text{active}) = \frac{f(D_A) \Pr(\text{active} \mid D_A)}{\Pr(\text{active})} = \frac{\frac{D_A}{D}}{\frac{1}{2}} = \frac{2D_A}{(D)^2},
$$

which completes the proof.

Proof of Lemma 2

If $D$ is small enough to satisfy $D < D_h(q, \hat{v}, D)$ which implies $I > 0$ given $(\hat{v}, q)$, then period 2 opens.

The retailer maximizes its 2nd period’s revenue (6) given (5). Suppose that $I \leq (\bar{\nu} - s_h) D$. Let $s_h$ denote the sale price to satisfy $I = (\bar{\nu} - s_h) D$ and $s_h \geq \hat{v}$. That is, the retailer has the inventory which it can sell to part of active consumers.

Differentiating (6) with respect to $s$ when $I > (\bar{\nu} - s_h) D$ gives

$$
\frac{dR(s,T)}{ds} = -2 \left( s - \frac{\bar{\nu}}{2} \right) D,
$$

which is negative under the assumption (8). This is because $s \geq \hat{v} \geq \underline{\nu} \geq \frac{\bar{\nu}}{2}$.

Thus, the retailer can increase (6) by lowering $s$ as much as possible. Thus, maximization of (6) holds when $I = (\bar{\nu} - s_h) D$, which is equivalent to

$$
s_h = \frac{(D - q)}{D} + \hat{v}.
$$

(35)
Note that (35) is valid when \( s_h \geq \hat{v} \) which is interchangeable with \( D \geq D_m (q, \hat{v}) (= q) \).

Suppose that \( I > (\overline{v} - \hat{v}) D \). That is, there are the inventory which the retailer can sell to all active consumers with \( v^2_c \geq \hat{v} \) and part of bargain-hunters.

The retailer has two choices: One is to sell its inventory only to active consumers at the sale price equal to \( s_m = b v \), and to dispose of the leftover \( I - (\overline{v} - \hat{v}) D \). Note that the retailer has no incentive to set its sale price such that \( v_B < s < \hat{v} \) since such pricing only reduces \( R (s, I) \). The other is to sell it to both active consumers and part of bargain-hunters at the sale price equal to \( s_l = v_B \), and to sell out of \( I \).

By the former choice, the retailer obtains its revenue given by
\[
\hat{v} (\overline{v} - \hat{v}) D,
\]
while by the latter choice, it obtains
\[
v_B (q - (\hat{v} - \overline{v}) D).
\]

From (36) and (37) with (7), we find that
\[
\hat{v} (\overline{v} - \hat{v}) D \geq v_B (q - (\hat{v} - \overline{v}) D)
\]
\[
\iff D \geq D_l (q, \hat{v}) = \frac{v_B q}{v_B + (\overline{v} - \hat{v}) (\hat{v} - v_B)},
\]
which completes the proof.

**Proof of Lemma 3**

Define \( \hat{v}_h \) such as \( \hat{v}_h = \overline{v} + \frac{q}{D} \) (\( \iff D_h (\hat{v}_h) = D \)). Then (13) is represented as
\[
EU_1 (\hat{v}; q) = \begin{cases} 
(v_M - p) \frac{q}{(\overline{v} - \hat{v}) D} \left[ 2 - \frac{q}{(\overline{v} - \hat{v}) D} \right] & \text{if } \hat{v}_h \leq \hat{v} \leq \overline{v}, \\
(v_M - p) & \text{if } \overline{v} \leq \hat{v} \leq \hat{v}_h.
\end{cases}
\]

Differentiation of \( EU_1 (\hat{v}; q) \) with respect to \( \hat{v} \) yields
\[
\frac{dEU_1 (\hat{v}; q)}{d\hat{v}} = \begin{cases} 
-2 (v_M - p) \left( 1 - \frac{q}{(\overline{v} - \hat{v}) D} \right) & \text{if } \hat{v}_h \leq \hat{v} < \overline{v}, \\
0 & \text{if } \overline{v} < \hat{v} \leq \hat{v}_h.
\end{cases}
\]

Note that \( EU_1 (\hat{v}; q) \) is differentiable at \( \hat{v} = \hat{v}_h \) and the border conditions are given as
\[
EU_1 (\hat{v}; q) = v_M - p > (v_M - p) \frac{q}{D} \left[ 2 - \left( \frac{q}{D} \right) \right] = EU_1 (\overline{v}; q).
\]
Thus, we find that (13) is a nonincreasing function of $\hat{v}$.

On the other hand, from (15), differentiation of $EU_2(\hat{v}; q)$ with respect to $\hat{v}$ yields

$$
\frac{dEU_2(\hat{v}; q)}{d\hat{v}} = \frac{1}{(D)^2} \left[ \frac{v_B q}{v_B + (\bar{v} - \hat{v})(\hat{v} - v_B)} \right]^2 \left[ \frac{v_B + (\hat{v} - v_B) (3\hat{v} - 2v_B - \bar{v})}{v_B + (\bar{v} - \hat{v})(\hat{v} - v_B)} \right],
$$

and the border conditions are given as

$$
EU_2(\hat{v}; q) = (\bar{v} - v_B) \left( \frac{v_B q}{D} \right)^2 < (\bar{v} - v_B) \left( \frac{q}{D} \right)^2 = EU_2(\bar{v}; q).
$$

If (18) and (7) hold, then (38) is positive since the next relationship is satisfied:

$$
3\hat{v} - 2v_B - \bar{v} \geq 3\bar{v} - 2v_B - \bar{v} = 2(\bar{v} - v_B) - 1 > 0.
$$

Thus, we find that (15) is an increasing function of $\hat{v}$.

We also find that, if $q$ is small enough to satisfy

$$
q \leq \frac{2D}{1 + (\frac{\bar{v} - v_B}{v_M - p})},
$$

then $EU_2(\bar{v}; q) \leq EU_1(\bar{v}; q)$, and that if $q$ is large enough to satisfy

$$
q \geq D \left( \frac{\bar{v}}{v_B} \right) \sqrt{\frac{v_M - p}{\bar{v} - v_B}},
$$

then $EU_1(\bar{v}; q) \leq EU_2(\bar{v}; q)$.

Note that the following inequalities hold

$$
\frac{2D}{1 + (\frac{\bar{v} - v_B}{v_M - p})} < D < D \left( \frac{\bar{v}}{v_B} \right) \sqrt{\frac{v_M - p}{\bar{v} - v_B}},
$$
given (3), (16) and (17). Therefore, the inequality,

$$
\frac{2D}{1 + (\frac{\bar{v} - v_B}{v_M - p})} < q < D \left( \frac{\bar{v}}{v_B} \right) \sqrt{\frac{v_M - p}{\bar{v} - v_B}},
$$

(41)
is well-defined. Combining this fact together with the properties about $EU_1 (\tilde{v}; q)$ and $EU_2 (\tilde{v}; q)$ derived above, we find that there exists the unique $\tilde{v} = \tilde{v}^* \in [\underline{v}, \overline{v}]$ to satisfy

$$EU_1 (\tilde{v}^*; q) = EU_2 (\tilde{v}^*; q)$$

when (41) holds.

References


Figure 3