Opening a Direct Digital Channel: 
the Impact of Versioning on the Physical Product 
Market with Heterogeneous Retailers 

Yuji Nakayama 

Discussion Paper New Series No. 2013-4 
March 2013 

School of Economics 

Osaka Prefecture University 

Sakai, Osaka 599-8531, Japan
Opening a Direct Digital Channel: 
the Impact of Versioning on the Physical Product 
Market with Heterogeneous Retailers*

Yuji Nakayama†
School of Economics, Osaka Prefecture University
March 29, 2013

Abstract

In this paper, we construct a vertical differentiation model comprising an upstream manufacturer and two downstream retailers with cost asymmetry. In this model, the manufacturer not only produces a physical product it sells to the downstream retailers, but also has an option of “versioning” to open a new direct channel for an alternate digital product. We find that the direct digital channel may reduce the quantity of the physical product sold by the inefficient retailer even if it increases total quantity. It may also increase the quantity of the physical product sold by the efficient retailer even if it reduces total quantity. Cost asymmetry across the retailers plays a role in these results. Moreover, under certain conditions related to the manufacturer’s cost and if the quality of the digital product is sufficiently low, then the manufacturer raises the wholesale price of the physical product and ceases to deal with the inefficient retailer, thereby eliminating retail competition, which may raise its retail price. This lowers the welfare of consumers who continue to purchase the physical product after the new digital product comes onto the market.

Keywords: versioning, vertical product differentiation, digital product, direct channel
JEL classification: L13, L15, L86

---

*This paper is based on part of my doctoral thesis, Nakayama (2013), presented to Osaka Prefecture University. I am grateful to my thesis committee members, namely, Nagateru Araki, Hiroyuki Morita, and Tatsuhiro Shichijo. I also would like to thank Hiroshi Uno, and the participants of a seminar at Fukushima University, especially Takaki Abe, Masakatsu Nakamura, and Toshihiro Sato for their helpful comments on its earlier version. All errors herein are of course my own. Financial support from the Japan Society for the Promotion of Science (JSPS) through a Grant-in-Aid for Scientific Research (C) (No. 23530280) is gratefully acknowledged.

†E-mail: nakayama@eco.osakafu-u.ac.jp
1 Introduction

The emergence of a new marketing channel affects the economy by expanding the consumer’s choice of products, altering the competitiveness of retail markets and affecting manufacturers’ profitability positively or negatively depending on the market structure. The emerging electronic commerce (e-commerce) channel, which operates through the Internet, constitutes a typical marketing channel with these features. Since its development in the second half of the 1990s, e-commerce has grown rapidly.

According to Japan’s Ministry of Economy, Trade and Industry (METI), in 2011, the size of Japanese businesses to consumer e-commerce market is about 8.5 trillion yen (METI, 2012). With an increased volume of transactions and a greater variety of items traded, today’s market is 8.6 percentage points larger than in 2010, and is 5.7 times larger than it was in 2001. In addition to ordinary retail sales, this figure includes sales of services such as hotel and restaurant reservations made over the Web and fees for Internet banking. The size of the retail and services market through e-commerce in 2011 was 12.2% higher than in the year before, at about 5.9 trillion yen. General merchandise retailing accounted for the largest share of retail and services e-commerce market (21.1% in 2011). However, the percentage of e-commerce market in total market, including traditional retailing, remains small. METI (2012) reports that, in 2011, e-commerce accounted for 2.83% of total retail and services market and 4.74% of general merchandise retailing. Nevertheless, it is widely believed that the potential economic impact of e-commerce is enormous.

In particular, the downloading of digital products directly from the Web has recently attracted attention because this practice has fundamentally transformed the content of businesses in such industries as music, publishing and software. According to the Recording Industry Association of Japan (RIAJ, 2012), Japanese music market has declined significantly in recent years. Annual sales of physical products such as music CDs and DVDs peaked at about 607 billion yen in 1998 and have since declined steadily, falling to about 282 billion yen in 2011. Although digital products such as music ringtones for cell phones and downloadable music from the Web have become important for the music industry, the market for content for mobile devices has declined in recent years after peaking at about 79.9 billion yen in 2008. By contrast, the market for digital music downloaded from the Web has continued to grow since the early 2000s; despite a slight decline in 2010, sales in 2011 amounted to 12.6 billion yen, which is about 6.8 times the 2005 figure.

Taking the above economic background into account, in this paper we examine the impact of a digital product on the market for a physical product with a vertical differentiation model.
à la Mussa and Rosen (1978). In this model, a manufacturer not only produces a physical product it sells to two downstream retailers, but also has an option of “versioning” to open a new direct channel for an alternate digital product. We focus on the downstream retail market, in which asymmetric retailers operate. We investigate a two-stage game in which the manufacturer first chooses its strategy, before the retailers compete with each other.

Our model is closely related to that in Chiang, Chhajed and Hess (2003). They use the vertical differentiation approach to develop a model that incorporates products of different quality à la Mussa and Rosen (1978). They analyze competition between conventional retailers and a manufacturer that opens a direct marketing channel. Although they do not cite Mussa and Rosen (1978), the structure of the demand side in their model is the same as that of Mussa and Rosen (1978).

Here, we explain the similarities and differences between their model and ours by using the notation of Chiang, Chhajed and Hess (2003). There are two differentiated products: a high-quality product and a low-quality product. We fix the quality of the high- and low-quality products at unity and $\theta (< 1)$, respectively. Let $v$ denote a consumer’s marginal evaluation of quality. Each consumer is characterized by a value of $v$, which is uniformly distributed on $[0, 1]$ with unit density. A monopoly manufacturer produces both products. The monopolist sells the high-quality product to two retailers at a wholesale price of $w$, and sells the low-quality product to consumers at a direct price of $p_d$. Hereafter, we call the high-quality product *product r*, and call the low-quality product *product d*. The retail market for product $r$ is served by a Cournot duopoly in which the retail price $p_r$ is determined by quantity competition between the two retailers. In these respects, the structure of our model is essentially the same as that of Chiang, Chhajed and Hess (2003, Section 6).

However, there are important differences of interpretation between our model and Chiang, Chhajed and Hess’s (2003). They consider a situation in which the same product is sold

---

1Mussa and Rosen (1978) is a useful model to analyze the competition of firms producing goods which has the difference of quality. This model has been often used and been extended in various directions for the subject of research in the literature of industrial organization and marketing science. See Avenel and Caprice (2006) and Toshimitsu (2008), for example. Price competition among firms is often assumed in the model, while there are papers in which quantity competition is used. Motta (1993), and Toshimitsu and Jinji (2007) derive equilibria of both price and quantity competition, and compare the two. Bacchisega, Randon and Zirulia (2012) is a recent example using the model.

2Belleflamme (2005) provides a unified framework to examine versioning strategies used in the information economy. Functional degradation analyzed in Csorba and Hahn (2006) is the theme related with this paper, although there is no network effect in our model. See also Deneckere and McAfee (1996). They show that a monopoly manufacturer can increase their profit by intentionally damaging a high-quality product to produce a low-quality counterpart even if the marginal cost of the latter is higher than that of the former.

3The horizontal differentiation approach à la Hotelling (1929) has also been used in the literature of industrial organization and marketing science. See Matsushima (2004), Coughlan and Soberman (2005), and Ishibashi and Matsushima (2009) for example. Recent researches on e-commerce using this approach include Nakayama (2009), Yoo and Lee (2011), and Vernik, Purohit and Desai (2011).
through two channels. The direct channel reduces the perceived quality of the product from one to \( \theta \), because consumers cannot examine the product before buying it. In addition, they impose a restriction on the manufacturer’s pricing such that \( w \leq p_d \). They do so to prevent the retailers from having an incentive to buy the product through the direct channel at the price \( p_d \) instead of buying it at the wholesale price \( w \). They show that opening a direct marketing channel enhances retail competitiveness, even if nothing is sold through this channel, and makes conventional retailers place more orders, which benefits the manufacturer. They also show that the profits of both the manufacturer and retailers increase after the direct channel opens if \( \theta \) falls in some range.\(^4\)

By contrast, we consider a situation in which product \( r \), which is sold through the retail channel, is physically different from product \( d \), which is sold through the direct digital channel via the Internet. For example, a computer software package in the form of a physical medium such as a CD-ROM is sold through the retail channel, and the same digital content is sold through the direct digital channel.\(^5\) In Chiang, Chhajed and Hess’s (2003) setting, computer software is sold through the retail and direct channels. By contrast, in our model, the two products differ from each other. Thus, there is no need to assume that retailers cannot purchase digital content through the direct channel because the digital product cannot be resold at physical stores. Thus, we allow the manufacturer to set \( p_d \) below \( w \). Moreover, we incorporate a new feature into the model. Let \( c_i (i = H, L) \) denote Retailer \( i \)’s marginal cost of handling the product. We introduce asymmetrical marginal costs: \( c_H > c_L \). Hereafter, the retailer with marginal costs of \( c_H \) (resp. \( c_L \)) is termed Retailer \( H \) (resp. Retailer \( L \)). Note that Chiang, Chhajed and Hess (2003) assume that \( c_i = 0 \).

This paper is organized as follows. In Section 2, we construct a vertical differentiation model that incorporates the features described above. Having first examined the equilibrium of a model that does not incorporate a direct digital channel, we derive the equilibrium of a model that does incorporate such a channel. In Section 3, we compare the equilibria derived in the previous section, focusing on the retail quantity and price. Section 4 contains concluding remarks. All proofs of lemmas and propositions are in the Appendix at the end of this paper.

\(^4\)Aiura (2007) uses a model similar to Chiang, Chhajed and Hess (2003), and analyzes the manufacturer’s wholesale pricing decision when one conventional retailer and one online retailer compete in the downstream market.

\(^5\)An alternative scenario for our model is that the manufacturer, rather than selling product \( d \) directly to consumers via the digital channel, sells it to consumers through multiple online platforms of digital contents. Because the platforms deal in the same content, their prices fall to the wholesale price set by the manufacturer.
2 Model

In this section, we first examine a model that does not incorporate a direct digital channel. Then, having examined one that incorporates such a channel, we compare the two models.

2.1 Model without a Direct Digital Channel

Consider a game played by a manufacturer and two retailers. The game proceeds as follows. First, the manufacturer determines the wholesale price $w$. Then, the retailers simultaneously choose their quantities $(q_H, q_L)$ given $w$. Consumers then decide whether to buy the product given the retail price $p_r$. We analyze this game and derive its subgame perfect equilibrium by backward induction.

2.1.1 Consumers

Consumer $v$’s surplus from product $r$ is

$$U^v_r = v - p_r.$$  \hfill (1)

Each consumer buys at most one unit of the product. If consumer $v_{\phi 1}$ is indifferent between buying product $r$ and not buying it, then $v_{\phi 1}$ satisfies

$$v_{\phi 1} = p_r.$$  \hfill (2)

Consumers whose marginal quality evaluation is at least $v_{\phi 1}$ purchase product $r$. Thus, the demand for product $r$ is

$$D_r = 1 - p_r \text{ if } 0 \leq p_r \leq 1.$$  \hfill (3)

2.1.2 Retailers

Retailer $i$ ($= H, L$) chooses quantity $q_i$ to maximize profit given the other retailer’s quantity $q_j$ ($j \neq i$) and the wholesale price, $w$, set by the manufacturer. Inverting (3) yields

$$p_r = 1 - D_r \text{ if } 0 \leq D_r \leq 1.$$  \hfill (4)

By using (4) and the market-clearing condition, $D_r = q_H + q_L$, we obtain the profit of
Retailer \( i \) as follows:

\[
\Pi'_i = (p_r - w - c_i) q_i = (1 - q_j - q_i - w - c_i) q_i \quad \text{if} \quad 0 \leq q_i < 1 - q_j, \quad (5)
\]

under the condition that \( 1 - q_j > 0 \). This condition means that Retailer \( j \) does not completely satisfy consumer demand.

Maximizing (5) with respect to \( q_i \) gives

\[
q_i = \begin{cases} 
0 & \text{if} \quad q_j > 1 - w - c_i, \\
\frac{1 - w - c_i - q_j}{2} & \text{if} \quad 0 \leq q_j \leq 1 - w - c_i. 
\end{cases} \quad (6)
\]

The following lemma shows the equilibrium quantities given \( w \):

**Lemma 1** In the subgame of retail competition in the absence of a direct digital channel, the equilibrium quantities given \( w \) are

\[
(q_H, q_L) = \begin{cases} 
(0, \frac{1 - w - c_L}{2}) & \text{if} \quad 1 + c_L - 2c_H \leq w < 1 - c_L, \\
\left( \frac{1 + c_L - 2c_H - w}{3}, \frac{1 + c_H - 2c_L - w}{3} \right) & \text{if} \quad w < 1 + c_L - 2c_H. 
\end{cases} \quad (7)
\]

Total quantity in the retail market, \( Q_r = q_H + q_L \), as a function of \( w \), is

\[
Q_r = \begin{cases} 
\frac{1 - w - c_L}{2} & \text{if} \quad 1 + c_L - 2c_H \leq w < 1 - c_L, \\
\frac{1 + c_L - 2c_H - w}{3} + \frac{1 + c_H - 2c_L - w}{3} & \text{if} \quad 0 \leq w \leq 1 + c_L - 2c_H. 
\end{cases} \quad (8)
\]

This lemma indicates that Retailer \( H \)’s quantity is zero when \( w \) exceeds \( 1 + c_L - 2c_H \). In addition, (8) has a kink at \( w = 1 + c_L - 2c_H \); we also have \( \frac{\partial Q_r}{\partial w} = -\frac{1}{2} \) if \( w > 1 + c_L - 2c_H \), while \( \frac{\partial Q_r}{\partial w} = -\frac{2}{3} \) if \( w < 1 + c_L - 2c_H \).

**2.1.3 Manufacturer**

The manufacturer chooses wholesale price \( w \) to maximize profit, taking (8) into account. The profit of the manufacturer is

\[
\Pi^m = (w - c_r) Q_r, \quad (9)
\]

where \( c_r \) is the marginal cost of producing product \( r \). There may exist two local maxima in (9) given the kink in \( Q_r \) at \( w = 1 + c_L - 2c_H \). We must take this into account when we consider the manufacturer’s maximization. The following lemma shows the profit-maximizing wholesale price set by the manufacturer when there is no direct channel:
Lemma 2 In the game in which there is no direct digital channel, the wholesale price $w$ that maximizes the manufacturer’s profit (9) is

$$w = \begin{cases} 
\frac{1}{2} c_r - \frac{1}{2} c_L + \frac{1}{2} & \text{if} \quad 1 - c_L - (\sqrt{3} + 2) (c_H - c_L) \leq c_r < 1 - c_L, \\
\frac{1}{2} c_r - \frac{1}{4} c_L - \frac{1}{4} c_H + \frac{1}{2} & \text{if} \quad c_r \leq 1 - c_L - (\sqrt{3} + 2) (c_H - c_L). 
\end{cases}$$

Note that if $c_r \leq 1 - c_L - (\sqrt{3} + 2) (c_H - c_L)$, then the wholesale price set by the manufacturer is low enough for the two retailers to be able to purchase the product. Otherwise, the high-cost retailer (i.e., Retailer $H$) cannot procure the product because the wholesale price is too high. If $c_r = 1 - c_L - (\sqrt{3} + 2) (c_H - c_L)$, then both $w = \frac{1}{2} c_r - \frac{1}{2} c_L + \frac{1}{2}$ and $w = \frac{1}{2} c_r - \frac{1}{4} c_L - \frac{1}{4} c_H + \frac{1}{2}$ maximize the manufacturer’s profit. That is, in this case, it is optimal for the manufacturer to sell to both retailers or to sell to only the low-cost retailer.

In what follows, we restrict our attention to the case in which both retailers are active. We assume that the following condition is satisfied:

Assumption 1

$$0 \leq c_r < 1 - c_L - (\sqrt{3} + 2) (c_H - c_L).$$

The right-hand side of the above inequality is positive if $c_L < 1$ and if $c_H$ is not too high relative to $c_L$. Then, we obtain the following proposition, in which the superscript $^*$ denotes an equilibrium outcome from the model that has no direct channel:

Proposition 1 Suppose that Assumption 1 holds. Then, in the absence of a direct digital channel, the equilibrium outcomes are

$$p_r^* = \frac{2 (2 + c_r) + (c_H + c_L)}{6}, \quad w^* = \frac{2 (1 + c_r) - (c_H + c_L)}{4},$$

$$q_H^* = \frac{2 (1 - c_r) - 7 c_H + 5 c_L}{12}, \quad q_L^* = \frac{2 (1 - c_r) + 5 c_H - 7 c_L}{12},$$

$$Q_r^* = \frac{2 (1 - c_r) - (c_H + c_L)}{6},$$

$$\Pi_H^* = \frac{(2 - 2 c_r - 7 c_H + 5 c_L)^2}{144}, \quad \Pi_L^* = \frac{(2 - 2 c_r + 5 c_H - 7 c_L)^2}{144},$$

$$\Pi^m = \frac{[2 (1 - c_r) - (c_H + c_L)]^2}{24}.$$
2.2 Model with a Direct Digital Channel

We add a direct digital marketing channel to the model and analyze a new game played by the manufacturer and the two retailers. This game proceeds as follows. First, the manufacturer determines the wholesale price \( w \) and the direct-channel price \( p_d \). Then, the retailers simultaneously choose their quantities given \( w \) and \( p_d \). Given \( p_r \) and \( p_d \), consumers then make their purchasing decisions. We analyze this game and derive its subgame perfect equilibrium by backward induction.

2.2.1 Consumers

Consumer \( v \)'s surplus from product \( d \) is

\[
U_v^d = \theta v - p_d. \tag{10}
\]

The surplus from product \( r \) is given by (1). Each consumer has three choices: (a) to buy product \( r \); (b) to buy product \( d \); or (c) to buy neither product. If consumer \( \hat{v} \) is indifferent between buying product \( r \) and buying product \( d \), then \( \hat{v} \) satisfies

\[
\hat{v} - p_r = \hat{v} - p_d \iff \hat{v} = \frac{p_r - p_d}{1 - \theta}.
\]

\( U_r^v \geq U_d^v \) if and only if \( v \geq \hat{v} \). If consumer \( v_{\phi 2} \) is indifferent between buying product \( d \) and not buying it, then \( v_{\phi 2} \) satisfies

\[
v_{\phi 2} \theta - p_d = 0 \iff v_{\phi 2} = \frac{p_d}{\theta}.
\]

The demand for product \( r \) is

\[
D_r = \begin{cases} 
1 - p_r & \text{if } p_r < \frac{p_d}{\theta}, \\
1 - \frac{p_r - p_d}{1 - \theta} & \text{if } \frac{p_d}{\theta} \leq p_r \leq 1 - \theta + p_d, \\
0 & \text{if } p_r > 1 - \theta + p_d.
\end{cases} \tag{11}
\]

The demand for product \( d \) is as follows:

\[
D_d = \begin{cases} 
0 & \text{if } p_r < \frac{p_d}{\theta}, \\
\frac{p_r - p_d}{1 - \theta} - \frac{p_d}{\theta} & \text{if } \frac{p_d}{\theta} \leq p_r \leq 1 - \theta + p_d, \\
1 - \frac{p_d}{\theta} & \text{if } p_r > 1 - \theta + p_d.
\end{cases} \tag{12}
\]

\footnote{Our presentation of equations (11) and (12) follows Aiura (2007), who comprehensively explains the demand structure of Chiang, Chhajed and Hess’s (2003) model.}
2.2.2 Retailers

Retailer \( i (= H, L) \) chooses quantity \( q_i \) to maximize profit given the other retailer’s quantity \( q_j \ (j \neq i) \), the wholesale price \( w \) and the direct price \( p_d \) set by the manufacturer.

Inverting (11) yields

\[
p_r = \begin{cases} 
1 - D_r & \text{if } 1 - \frac{p_d}{\theta} < D_r < 1, \\
1 - (\theta - p_d) - (1 - \theta) D_r & \text{if } 0 \leq D_r \leq 1 - \frac{p_d}{\theta}, 
\end{cases} 
\tag{13}
\]

under the condition that \( \theta > p_d \), which means that some consumers whose quality evaluation is high derive a positive surplus from consuming product \( d \). If \( \theta < p_d \), no consumer is willing to buy product \( d \). Thus, the outcome is the same as that in the absence of a direct channel.

In what follows, we assume \( \theta > p_d \).

Note that (13) has a kink at \( D_r = 1 - \frac{p_d}{\theta} \). This kink generates discontinuity in the retailers’ marginal revenue. We take this into account when considering the retailers’ profit maximization. By using (13) and the market-clearing condition, \( D_r = q_H + q_L \), we obtain the profit of Retailer \( i \) as follows:

\[
\Pi'_r = (p_r - w - c_i) q_i = \begin{cases} 
(1 - q_j - q_i - w - c_i) q_i & \text{if } 1 - q_j - \frac{p_d}{\theta} < q_i < 1 - q_j, \\
[(1 - \theta)(1 - q_j - q_i) + p_d - w - c_i] q_i & \text{if } 0 \leq q_i \leq 1 - q_j - \frac{p_d}{\theta}. 
\end{cases} 
\tag{14}
\]

Note that (14) is derived under the condition, \( 1 - q_j - \frac{p_d}{\theta} > 0 \), which means that the demand for product \( d \) is positive when \( q_i = 0 \). If \( q_i \) is large enough to ensure that \( 1 - q_j - \frac{p_d}{\theta} < q_i \), then \( p_r \) is low enough to make product \( d \)'s demand zero. If \( q_i \) is so small that \( 1 - q_j - \frac{p_d}{\theta} \geq q_i \), then demand for product \( d \) is positive. Thus, Retailer \( i \)'s profit depends on the level of \( q_i \).

By maximizing (14) with respect to \( q_i \), we obtain the following expression for Retailer \( i \)'s optimal quantity:

\[
q_i = \begin{cases} 
0 & \text{if } w + c_i \geq 1 - \theta + p_d - (1 - \theta) q_j, \\
\frac{1 - \theta + p_d - w - c_i - (1 - \theta) q_j}{2(1 - \theta)} & \text{if } (1 - \theta)(q_j - 1) + \left(\frac{2}{\theta} - 1\right) p_d \leq w + c_i < \theta - p_d - (1 - \theta) q_j, \\
1 - q_j - \frac{p_d}{\theta} & \text{if } q_j + \frac{2}{\theta} p_d - 1 \leq w + c_i \leq (1 - \theta)(q_j - 1) + \left(\frac{2}{\theta} - 1\right) p_d, \\
\frac{1 - w - c_i - q_j}{2} & \text{if } 0 \leq w + c_i < q_j + \frac{2}{\theta} p_d - 1, 
\end{cases} 
\tag{15}
\]
which depends on Retailer $i$’s marginal cost, $w + c_i$. Note that (15) is derived under the condition that $q_j + \frac{2}{\theta} p_d - 1 > 0$. Note also that Retailer $i$’s marginal revenue approaches $q_j + \frac{2}{\theta} p_d - 1$ as $q_i$ approaches $1 - q_j - \frac{p_d}{\theta}$ from above; when $q_i$ approaches $1 - q_j - \frac{p_d}{\theta}$ from below, its marginal revenue is $(1 - \theta) (q_j - 1) + \left( \frac{2}{\theta} - 1 \right) p_d$. Notice that

$$
(1 - \theta) (q_j - 1) + \left( \frac{2}{\theta} - 1 \right) p_d - \left( q_j + \frac{2}{\theta} p_d - 1 \right)
$$

$$
= \theta \left( 1 - q_j - \frac{p_d}{\theta} \right) > 0.
$$

That is, as previously pointed out, there is discontinuity in the retailer’s marginal revenue. If $w$ satisfies the inequality

$$
q_j + \frac{2}{\theta} p_d - 1 \leq w + c_i \leq (1 - \theta) (q_j - 1) + \left( \frac{2}{\theta} - 1 \right) p_d,
$$

then Retailer $i$’s optimal quantity does not depend on its marginal cost $w + c_i$.\(^7\)

The following lemma shows the equilibrium quantities given $w$:

**Lemma 3** In the subgame of retail competition in the presence of a direct digital channel, the equilibrium quantities given $(p_d, w)$ are as follows:

\[
(q_H, q_L) = \begin{cases} 
\left( \frac{1 - \theta + c_L - 2c_H + p_d - w}{3(1 - \theta)}, \frac{1 - \theta + c_H - 2c_L - p_d - w}{3(1 - \theta)} \right) & \text{if } (p_d, w) \in R_{1a}, \\
\left( 0, \frac{1 - \theta - c_L + p_d - w}{2(1 - \theta)} \right) & \text{if } (p_d, w) \in R_{1b}, \\
\text{any combination satisfying } q_H + q_L = 1 - \frac{p_d}{\theta} & \text{if } (p_d, w) \in R_2, \\
\left( \frac{1 + c_L - 2c_H - w}{3}, \frac{1 + c_H - 2c_L - w}{3} \right) & \text{if } (p_d, w) \in R_{3a}, \\
\left( 0, \frac{1 - w - c_L}{2} \right) & \text{if } (p_d, w) \in R_{3b},
\end{cases}
\]  

\(^7\)If $q_j + \frac{2}{\theta} p_d - 1 > 0$, then $(1 - \theta) (q_j - 1) + \left( \frac{2}{\theta} - 1 \right) p_d > 0$. Thus, (15) is valid under the former inequality. If either or both inequalities do not hold, then part of (15) no longer represents Retailer $i$’s optimal quantity. However, Lemma 3 is valid because it is conditioned on $p_d$ correspondingly.
where the regions conditioning \((p_d, w)\) are defined as

\[
R_{1a} = \begin{cases} 
(p_d, w) | w \leq 1 - \theta + c_L - 2c_H + p_d, & w \geq \frac{3 - \theta}{2\theta} - \frac{1 - \theta + c_H + c_L}{2} 
\end{cases},
\]

\[
R_{1b} = \begin{cases} 
(p_d, w) | w \leq 1 - \theta - c_L + p_d, & w \geq \frac{3 - \theta}{2\theta} - \frac{1 - \theta + c_H + c_L}{2} 
\end{cases},
\]

\[
R_2 = \begin{cases} 
(p_d, w) | w \leq \frac{3 - \theta}{2\theta} - \frac{1 - \theta + c_H + c_L}{2}, & w \geq \frac{3 - \theta}{2\theta} - \frac{1 - \theta + c_H + c_L}{2} 
\end{cases},
\]

\[
R_{3a} = \begin{cases} 
(p_d, w) | w \leq 1 + c_L - 2c_H, & w \leq \frac{3 - \theta}{2\theta} - \frac{1 + c_H + c_L}{2} 
\end{cases},
\]

\[
R_{3b} = \begin{cases} 
(p_d, w) | w \leq 1 - c_L, & w \geq 1 + c_L - 2c_H, & w \leq \frac{2}{\theta} - p_d - (1 + c_L) 
\end{cases}.
\]

The regions defined in Lemma 3 are illustrated in Figure 1, which is based on the following parameter values:

\[
\theta = 0.5, \quad c_H = 0.1, \quad c_L = 0.05. \quad (17)
\]

The coordinates of points A–G in this figure are as follows:

- Point A \((\theta, 1 - c_L)\),
- Point B \((\theta(1 - c_H + c_L), 1 + c_L - 2c_H)\),
- Point C \((\frac{\theta(1 - \theta - c_H + c_L)}{1 - \theta}, \frac{1 - \theta - (2 - \theta)c_H + c_L}{1 - \theta})\),
- Point D \((\frac{\theta(1 + c_H + c_L)}{3}, 0)\),
- Point E \((\frac{\theta(1 - \theta + c_H + c_L)}{3 - \theta}, 0)\),
- Point F \((0, 1 - \theta - c_L)\),
- Point G \((0, 1 - \theta + c_L - 2c_H)\).

If \(c_H = c_L = 0\), regions \(R_{1b}\) and \(R_{3b}\) are nonexistent, and the figure is the same as Figure 4 in Chiang, Chhajed and Hess (2003), except that the line along which \(w = p_d\) is omitted from Chiang, Chhajed and Hess’s (2003) figure.

Total output in the retail market as a function of \(p_d\) and \(w\) is summarized as follows:

\[
Q_r = \begin{cases} 
\frac{1 - \theta + c_L - 2c_H + p_d - w}{2(1 - \theta)} & \text{if } (p_d, w) \in R_{1a}, \\
\frac{1 - \theta + c_L + p_d - w}{2(1 - \theta)} & \text{if } (p_d, w) \in R_{1b}, \\
\frac{1}{\theta} - \frac{p_d}{\theta} & \text{if } (p_d, w) \in R_2, \\
\frac{1 + c_L - 2c_H - w}{3} & \text{if } (p_d, w) \in R_{3a}, \\
\frac{1 - w - c_L}{2} & \text{if } (p_d, w) \in R_{3b}.
\end{cases} \quad (18)
\]
The corresponding summary of retail price is as follows:

\[
p_r = \begin{cases} 
  \frac{1-\theta+c_L+2w}{3} + \frac{1-\theta+c_H+p_d-w}{3(1-\theta)} & \text{if } (p_d, w) \in R_{1a}, \\
  \frac{1-\theta+c_L+w}{2} + \frac{1-\theta+c_H+p_d-w}{3(1-\theta)} & \text{if } (p_d, w) \in R_{1b}, \\
  \frac{p_d}{\theta} & \text{if } (p_d, w) \in R_2, \\
  \frac{1+c_L+c_H+2w}{3} & \text{if } (p_d, w) \in R_{3a}, \\
  \frac{1+c_L+w}{2} & \text{if } (p_d, w) \in R_{3b}.
\end{cases}
\]  

(19)

Note that demand for product \(d\) is zero if \((p_d, w) \in R_{2} \cup R_{3a} \cup R_{3b}\). \(Q_d\) denotes the output for the direct channel. Note that \(Q_r\) and \(Q_d\) satisfy \(Q_r + Q_d = 1 - \frac{p_d}{\theta}\). Thus, \(Q_d\) is summarized as follows:

\[
Q_d = \begin{cases} 
  1 - \frac{p_d}{\theta} - \left(\frac{1-\theta+c_L-2c_H+p_d-w}{3(1-\theta)} + \frac{1-\theta+c_H+2c_L+p_d-w}{3(1-\theta)}\right) & \text{if } \begin{array}{l} (p_d, w) \in R_{1a}, \\
\end{array} \\
  1 - \frac{p_d}{\theta} - \left(\frac{1-\theta+c_L+p_d-w}{2(1-\theta)}\right) & \text{if } \begin{array}{l} (p_d, w) \in R_{1b}, \\
\end{array} \\
  0 & \text{if } \begin{array}{l} (p_d, w) \in R_2 \cup R_{3a} \cup R_{3b}.
\end{array}
\end{cases}
\]  

(20)
2.2.3 Manufacturer

The manufacturer chooses the direct price $p_d$ and the wholesale price $w$ to maximize profit. Given $Q_r$ and $Q_d$ as (18) and (20), respectively, the manufacturer’s profit is

$$
\Pi_m = (w - c_r) Q_r + (p_d - c_d) Q_d,
$$

where $c_d$ is the marginal cost of producing product $d$.

Note that (18) and (20) have different shapes depending on the region in which the pair $(p_d, w)$ lies, and that neither is differentiable with respect to the price along the border between the two regions. Thus, there may exist different local maxima in (21). We take this into account when considering the manufacturer’s maximization. However, it is difficult to solve the maximization problem without imposing conditions. Hence, to simplify our analysis, we make the following assumption:

**Assumption 2**

$$
0 \leq c_r < 1 - c_L - \frac{3 (\sqrt{3} + 2) - (\sqrt{3} + 3) \theta}{3 (1 - \theta)} (c_H - c_L).
$$

The right-hand side of the above inequality is positive if $c_L < 1$ and if $c_H$ is not too large relative to $c_L$. Note that making Assumption 2 is sufficient to ensure that Assumption 1 holds. This is because the following relationship holds:

$$
\frac{3 (\sqrt{3} + 2) - (\sqrt{3} + 3) \theta}{3 (1 - \theta)} - (\sqrt{3} + 2) = \frac{\theta}{3 (1 - \theta)} \left(2 \sqrt{3} + 3\right) > 0.
$$

The following lemma shows the profit-maximizing price set by the manufacturer operating with a direct channel:

**Lemma 4** Suppose that Assumption 2 holds. Then, the pair of prices (direct-channel and wholesale), $(p_d, w)$, that maximize (21) is

$$
(p_d, w) = \begin{cases} 
\left(\frac{\theta + c_d}{2}, \frac{2(1+c_r) - (c_H + c_L)}{4}\right) & \text{if } (c_d, c_r) \in \hat{R}_{1a}, \\
\left(\frac{\theta + c_d}{2}, \frac{1-c_r + c_c}{2}\right) & \text{if } (c_d, c_r) \in \hat{R}_{1b}, \\
\left(\frac{\theta(4-2\theta+c_H+c_L+2c_c)}{2(3-\theta)}, \frac{2(1+c_r) - (c_H + c_L)}{4}\right) & \text{if } (c_d, c_r) \in \hat{R}_2,
\end{cases}
$$

13
where the regions conditioning \((c_d, c_r)\) are defined as

\[
\tilde{R}_{1a} = \begin{cases} (c_d, c_r) & c_d < \theta \frac{1 - \theta + c_H + c_L + 2c_r}{3 - \theta}, \\ c_d \geq c_r + c_L + (\sqrt{3} + 2) (c_H - c_L) - (1 - \theta), & \end{cases}
\]

\[
\tilde{R}_{1b} = \begin{cases} (c_d, c_r) & c_d \leq c_r + c_L + (\sqrt{3} + 2) (c_H - c_L) - (1 - \theta), \\ c_r < 1 - c_L - \frac{3(\sqrt{3} + 2 - (\sqrt{3} + 3)\theta)}{3(1 - \theta)} (c_H - c_L), & \end{cases}
\]

\[
\tilde{R}_{2} = \begin{cases} (c_d, c_r) & c_d \geq \theta \frac{1 - \theta + c_H + c_L + 2c_r}{3 - \theta}, \\ c_r < 1 - c_L - \frac{3(\sqrt{3} + 2 - (\sqrt{3} + 3)\theta)}{3(1 - \theta)} (c_H - c_L). & \end{cases}
\]

Figure 2 shows the regions defined in Lemma 4. This figure is based on the parameter values given in (17), which were also used for Figure 1. The coordinates of points A–G in this figure are as follows:

- Point A \(\left( \theta \left[ 1 - \frac{2\sqrt{3} + 3}{3(1 - \theta)} (c_H - c_L) \right], 1 - c_L - \frac{3(\sqrt{3} + 2) - (\sqrt{3} + 3)\theta}{3(1 - \theta)} (c_H - c_L) \right)\),
- Point B \(\left( \theta \frac{1 - \theta + c_H + c_L}{3 - \theta}, 0 \right)\),
- Point C \(\left( 0, (1 - \theta) - c_L - (\sqrt{3} + 2) (c_H - c_L) \right)\),
- Point D \(\left( 0, 1 - \theta - c_L \right)\),
- Point E \(\left( \theta + \frac{c_H - c_r}{\theta - 3} \left[ 3\sqrt{3} + 6 - \theta (\sqrt{3} + 3) \right], 1 - c_L - \frac{3(\sqrt{3} + 2) - (\sqrt{3} + 3)\theta}{3(1 - \theta)} (c_H - c_L) \right)\).

If \((c_d, c_r) \in \tilde{R}_{1a} \cup \tilde{R}_{1b}\), then the manufacturer set the prices \((p_d, w)\) so that demand for both products is positive. When \(c_r\) is small enough to be included in \(\tilde{R}_{1a}\), a low wholesale price \((w = \frac{2(1 + c_r) - (c_H + c_L)}{4})\) is set to allow Retailers \(H\) and \(L\) to procure product \(r\) from the manufacturer. Otherwise, a high wholesale price \((w = \frac{1 - c_H + c_r}{2})\) is set to attract only Retailer \(L\). Note that if \((c_d, c_r)\) lies along segment AC in Figure 2, then both wholesale prices are optimal for the manufacturer. If \((c_d, c_r) \in \tilde{R}_2\), then the manufacturer sets \(p_d\) equal to \(\frac{\theta(4 - 2\theta + c_H + c_L + 2c_r)}{2(3 - \theta)}\), which is high enough to eliminate the demand for product \(d\). That is, consumers whose quality evaluation is \(v \geq v_{d2} = \frac{\theta d}{2}\) purchase product \(r\) and the rest purchase neither product. Note that this direct price does not depend on \(c_d\). Thus, the manufacturer maintains this direct price as long as \(c_d \geq \theta \frac{1 - \theta + c_H + c_L + 2c_r}{3 - \theta}\).

Given Lemma 4, we can derive the equilibrium outcomes for the three cases as functions of the costs \((c_d, c_r)\). We summarize the outcomes in the following proposition, in which the superscript ** denotes an equilibrium outcome from the model incorporating the direct channel:
Proposition 2 Suppose that Assumption 2 holds. Then, when there is a direct digital channel, the equilibrium outcomes are as follows:
1. if \((c_d, c_r) \in \bar{R}_{1a}\),

\[
\begin{align*}
    p_r^{**} &= \frac{4 - \theta + c_H + c_L + c_d + 2c_r}{6}, &
    p_d^{**} &= \frac{\theta + c_d}{2}, \\
    w^{**} &= \frac{2 (1 + c_r) - (c_H + c_L)}{4}, \\
    q_h^{**} &= \frac{2 (1 - \theta - c_L - c_r + c_d) - 7 (c_H - c_L)}{12 (1 - \theta)}, \\
    q_l^{**} &= \frac{2 (1 - \theta - c_r - c_L + c_d) + 5 (c_H - c_L)}{12 (1 - \theta)}, \\
    Q_r^{**} &= \frac{2 (1 - \theta - c_r + 2c_d - (c_H + c_L))}{6 (1 - \theta)}, \\
    Q_d^{**} &= \frac{\theta (1 - \theta + 2c_r + c_H + c_L) - (3 - \theta) c_d}{6 \theta (1 - \theta)}, \\
    \Pi_H^{**} &= \frac{(2 \theta + 7c_H - 5c_L - 2c_d + 2c_r - 2)^2}{144 (1 - \theta)}, \\
    \Pi_L^{**} &= \frac{[2 (1 - \theta) + 2 (c_d - c_r - c_L) + 5 (c_H - c_L)]^2}{144 (1 - \theta)}, \\
    \Pi_m^{**} &= \frac{(2 \theta + c_H + c_L - 2c_d + 2c_r - 2)^2}{24 (1 - \theta)} + \frac{(\theta - c_d)^2}{4 \theta};
\end{align*}
\]

2. if \((c_d, c_r) \in \bar{R}_{1b}\),

\[
\begin{align*}
    p_r^{**} &= \frac{3 - \theta + c_r + c_L + c_d}{4}, &
    p_d^{**} &= \frac{\theta + c_d}{2}, \\
    w^{**} &= \frac{1 - c_L + c_r}{2}, \\
    q_h^{**} &= 0, &
    q_l^{**} &= \frac{1 - \theta - c_L + c_d - c_r}{4 (1 - \theta)} (= Q_r^{**}), \\
    Q_d^{**} &= \frac{\theta (1 - \theta + c_r + c_L) - (2 - \theta) c_d}{4 \theta (1 - \theta)}, \\
    \Pi_H^{**} &= 0, &
    \Pi_L^{**} &= \frac{(1 - \theta - c_L + c_d - c_r)^2}{16 (1 - \theta)}, \\
    \Pi_m^{**} &= \frac{(1 - \theta - c_r + c_d - c_L)^2}{8 (1 - \theta)} + \frac{(\theta - c_d)^2}{4 \theta};
\end{align*}
\]
3. if \((c_d, c_r) \in \tilde{R}_2\),

\[
\begin{align*}
\bar{p}_r^* &= \frac{2(2 - \theta + c_L + c_r) + (c_H - c_L)}{2(3 - \theta)}, \\
\bar{p}_d^* &= \frac{\theta (4 - 2\theta + c_H + c_L + 2c_r)}{2(3 - \theta)}, \\
w^{**} &= \frac{2(1 + c_r) - (c_H + c_L)}{4}, \\
(q_H^{**}, q_L^{**}) : \text{any combination satisfying } q_H^{**} + q_L^{**} = 1 - \frac{p_d^{**}}{\theta}, \\
Q_r^{**} &= \frac{2(1 - c_r) - (c_H + c_L)}{2(3 - \theta)}, \\
Q_d^{**} &= 0, \\
\Pi_{rH}^{**} &= \frac{1}{1 - \theta} \left[ \frac{2(1 - \theta)(1 - c_r - c_L) - (7 - 3\theta)(c_H - c_L)}{4(3 - \theta)} \right]^2, \\
\Pi_{rL}^{**} &= \frac{1}{1 - \theta} \left[ \frac{2(1 - \theta)(1 - c_r - c_L) + (5 - \theta)(c_H - c_L)}{4(3 - \theta)} \right]^2, \\
\Pi_{m}^{**} &= \frac{[2 - 2c_r - (c_H + c_L)]^2}{8(3 - \theta)}.
\end{align*}
\]

3 Equilibrium Comparison

In this section, we examine the equilibrium outcomes derived in the previous section from the models incorporating and omitting a direct digital channel. Specifically, we compare the equilibrium outcomes by focusing on the retail quantity and price, and discuss the results.

3.1 Quantity

We compare the equilibrium quantities of product \(r\) in the models with and without a direct channel by using the results of Propositions 1 and 2. First, let us consider the total quantity \(Q_r\). We obtain the following proposition:

**Proposition 3** Suppose that Assumption 2 holds. Then, it follows that

1. if \((c_d, c_r) \in \tilde{R}_{1a}\),

\[
Q_r^{**} > Q_r^* \\
\iff c_r < \frac{1}{\theta} c_d - c_L - \frac{1}{2} (c_H - c_L); \quad (22)
\]
2. If \((c_d, c_r) \in \widehat{R}_{1b}\) with the condition \(\theta > 3 - \frac{3}{7}\sqrt{3} (\approx 0.402)\),

\[
Q_r^{**} > Q_r^* \\
\iff c_r < \frac{3}{4\theta - 1}c_d - c_L - \frac{1 - \theta}{4\theta - 1} [1 - 2 (c_H - c_L)];
\]

otherwise

\[
Q_r^{**} < Q_r^*;
\]

3. If \((c_d, c_r) \in \widehat{R}_2\),

\[
Q_r^{**} > Q_r^*.
\]

This proposition reveals a number of findings. First, given a \(c_r\) satisfying Assumption 2, if \(c_d\) is relatively large so that the pair \((c_d, c_r)\) lies in \(\widehat{R}_2\), then opening a direct digital channel for product \(d\) increases the total quantity of product \(r\). Note that, in this case, the demand for product \(d\) is zero and the wholesale price to the retailers does not change. Next, suppose that \(c_d\) takes an intermediate value so that the pair \((c_d, c_r)\) lies in \(\widehat{R}_{1a}\). Then, opening the direct digital channel increases the total quantity of product \(r\) if and only if \(c_r\) is small enough to satisfy (22). Next, suppose that \(c_d\) is small so that the pair \((c_d, c_r)\) lies in \(\widehat{R}_{1b}\). In addition, suppose that \(\theta > 3 - \frac{3}{7}\sqrt{3}\). Then, opening the direct digital channel increases the total quantity of product \(r\) if and only if \(c_r\) is small enough to satisfy (23). If \(\theta\) is large, product \(d\) is a close substitute for product \(r\). Facing the entry of such a product, Retailer \(L\) behaves competitively and increases the quantity if \(c_r\) is small. Note that, in this case, the wholesale price to the retailers increases, and as a result, Retailer \(H\)’s quantity falls to zero. By contrast, if \(\theta \leq 3 - \frac{3}{7}\sqrt{3}\), then opening the direct digital channel reduces the total quantity of product \(r\). If \(\theta\) is small, product \(d\) is a poor substitute for product \(r\). Thus, Retailer \(L\) behaves less competitively.

Next, consider the equilibrium quantities sold by each retailer, \(q_H\) and \(q_L\). From Proposition 2, in \(\widehat{R}_2\), these quantities are not uniquely determined because any combination satisfying \(q_h^{**} + q_l^{**} = 1 - \frac{\theta_{d_{**}}}{\theta}\) is an equilibrium outcome. We also know that \(q_H = 0\) in \(\widehat{R}_{1b}\). Thus, we restrict our attention to the case in which \((c_d, c_r) \in \widehat{R}_{1a}\). Hence, we obtain the following proposition:
Proposition 4 Suppose that Assumption 2 holds. If \((c_d, c_r) \in \hat{R}_{1a}\), then it follows that

\[
q_H^{**} > q_H^{*} \iff c_r < \frac{1}{\theta} c_d - c_L - \frac{7}{2} (c_H - c_L),
\]

\[
q_L^{**} > q_L^{*} \iff c_r < \frac{1}{\theta} c_d - c_L + \frac{5}{2} (c_H - c_L).
\]

From Propositions 3 and 4, we obtain a number of findings. If it is the case that

\[
\frac{1}{\theta} c_d - c_L - \frac{7}{2} (c_H - c_L) < c_r < \frac{1}{\theta} c_d - c_L - \frac{1}{2} (c_H - c_L),
\]

then opening the direct digital channel reduces the quantity sold by Retailer H even if it increases the total quantity of product r. By contrast, if it is the case that

\[
\frac{1}{\theta} c_d - c_L - \frac{1}{2} (c_H - c_L) < c_r < \frac{1}{\theta} c_d - c_L + \frac{5}{2} (c_H - c_L),
\]

then opening the direct digital channel increases the quantity sold by Retailer L even if it reduces the total quantity of product r. If there is no cost asymmetry between the two retailers (i.e., \(c_H = c_L\)), both the total quantity and each retailer’s quantity increase if and only if \(c_r < \frac{1}{\theta} c_d - c_L\). Thus, cost differences between retailers cause changes in total and
Figure 4: The Total Quantity with and without the Direct Channel when $c_d = 0.15$

Figure 3 uses the parameter values in (17) to illustrate the roles of the condition $(c_d, c_r) \in \hat{R}_{1a}$ in Propositions 3 and 4. The three dashed lines in this figure, along which the pair $(c_d, c_r)$ satisfies $q_{L}^{**} = q_{L}^{*}$, $Q_{r}^{**} = Q_{r}^{*}$ and $q_{H}^{**} = q_{H}^{*}$, from left to right, respectively, divide region $\hat{R}_{1a}$ (i.e., the polygon ABOC in this figure) into four parts. In the part of $\hat{R}_{1a}$ between $Q_{r}^{**} = Q_{r}^{*}$ and $q_{H}^{**} = q_{H}^{*}$, which satisfies (24), when the direct channel opens, the quantity sold by Retailer $H$ decreases, but the total quantity of product $r$ increases (i.e., $q_{H}^{**} < q_{H}^{*}$ and $Q_{r}^{**} > Q_{r}^{*}$). In the part of $\hat{R}_{1a}$ between $q_{L}^{**} = q_{L}^{*}$ and $Q_{r}^{**} = Q_{r}^{*}$, which satisfies (25), when the direct channel opens, the quantity sold by Retailer $L$ increases, but the total quantity of product $r$ decreases (i.e., $q_{L}^{**} > q_{L}^{*}$ and $Q_{r}^{**} < Q_{r}^{*}$). Note that a line representing $c_r = c_d$ is also included in this figure. So far, we have not specified conditions relating to $c_r$ and $c_d$ when deriving the equilibrium for the model incorporating the direct channel. Indeed, it is plausible to suppose that the marginal cost of the physical product exceeds that of its digital counterpart. There is a region above the line $c_r = c_d$ in $\hat{R}_{1a}$ that satisfies (24) and (25). That is, the total and individual quantities shift conversely when $c_r > c_d$. However, in no region does the quantity sold by Retailer $H$ increase following the opening of the direct channel when $c_r > c_d$ (at least given the parameter values in (17)).
Having added $c_d = 0.15$ to the parameter values given in (17), we draw Figures 4 and 5, which illustrate the resultant total and individual quantities. Figure 4 shows that the total quantity increases if $c_r < 0.225$. Figure 5 shows that the quantity sold by Retailer L increases if $c_r < 0.375$. Therefore, based on a wide range of parameter values for $c_r$, the quantity sold by Retailer L increases even if the total quantity of product $r$ decreases following the opening of the direct channel. Indeed, as Figure 3 indicates, there remain cases in which $q_{L}^{**} > q_{L}^{*}$ even if $c_d$ approaches zero.

Next, we consider the case in which $(c_d, c_r) \in \tilde{B}_{1b}$ in Propositions 3. The dashed line in Figure 6, along which the pair $(c_d, c_r)$ satisfies $Q_{r}^{**} = Q_{r}^{*}$, divides region $\tilde{B}_{1b}$ into two parts. This figure is based on (17). In the part of $\tilde{B}_{1b}$ below $Q_{r}^{**} = Q_{r}^{*}$, which satisfies (23), when the direct channel opens, the total quantity of product $r$ increases (i.e., $Q_{r}^{**} > Q_{r}^{*}$). Figure 6 shows that this part, in which $c_r > c_d$, is small. Thus, the finding that the opening of the direct channel increases the total quantity of product $r$ applies only if the restrictive condition on $(c_d, c_r)$ holds. The result does not apply if $c_d = 0$. In other words, we miss 2 of Proposition 3 if we assume that $c_d = 0$. Note that the dashed line indicating $Q_{r}^{**} = Q_{r}^{*}$ changes location to across or below Point A if $\theta \leq 3 - \frac{3}{2} \sqrt{3} (\approx 0.402)$. Thus, in $\tilde{B}_{1b}$, the total
quantity of product $r$ unambiguously decreases following the opening of the direct channel if $\theta$ is so small that $\theta \leq 3 - \frac{3}{7} \sqrt{3}$.

3.2 Price

In this subsection, we examine how the retail price of product $r$ changes following the opening of the direct digital channel. We obtain the following proposition:

**Proposition 5** Suppose that Assumption 2 holds. Then, it follows that

1. if $(c_d, c_r) \in \hat{R}_{1a}$,
   
   \[ p_r^{**} < p_r^*; \]

2. if $(c_d, c_r) \in \hat{R}_{1b}$ with the condition $\theta < \frac{21}{37} - \frac{6}{37} \sqrt{3} (\approx 0.287)$,
   
   \[ p_r^{**} > p_r^* \]
   \[ \Leftrightarrow c_r < 1 - 3\theta - c_L - 2(c_H - c_L) + 3c_d; \]  \hspace{1cm} (26)

otherwise

\[ p_r^{**} < p_r^*; \]
3. if \((c_d, c_r) \in \tilde{R}_2\),
\[ p_{r}^{**} < p_{r}^*. \]

A number of findings emerge from this proposition. Given a \(c_r\) satisfying Assumption 2, if the pair \((c_d, c_r)\) lies in \(\tilde{R}_{1a}\) or \(\tilde{R}_2\), then the opening of a direct digital channel for product \(d\) unambiguously reduces the retail price of product \(r\). This means that the introduction of product \(d\) enhances retail competition, and all consumers that purchase product \(r\) or product \(d\) obtain an additional surplus. This result also applies if the pair \((c_d, c_r)\) lies in \(\tilde{R}_{1b}\) provided \(\theta \geq \frac{21}{37} - \frac{6}{37}\sqrt{3}\). However, suppose that \(\theta < \frac{21}{37} - \frac{6}{37}\sqrt{3}\). Then, the retail price of product \(r\) increases when \(c_d\) is large enough to satisfy (26). Note that the wholesale price to the retailers increases, and only Retailer \(L\) operates in product \(r\)'s retail market, if the pair \((c_d, c_r)\) lies in \(\tilde{R}_{1b}\). If \(\theta\) is small, product \(d\) is a poor substitute for product \(r\). The entry of such a product leads Retailer \(L\) to behave less competitively. As a result, this new entry reduces the consumer surplus of consumers who continue to purchase product \(r\) following the introduction of product \(d\).

Figure 7 is based on the following parameter values:

\[ \theta = 0.2, \quad c_H = 0.1, \quad c_L = 0.05. \] (27)

The dashed line in Figure 7, along which the pair \((c_d, c_r)\) satisfies \(p_{r}^{**} = p_{r}^*\), divides the region \(\tilde{R}_{1b}\) into two parts. The part below \(p_{r}^{**} = p_{r}^*\) in \(\tilde{R}_{1b}\) satisfies (26). In this part, when the direct channel opens, the retail price of product \(r\) increases (i.e., \(p_{r}^{**} > p_{r}^*\)). Note that \(c_r > c_d\) in this part. Figure 7 shows that this part, in which \(c_r > c_d\), is small. Thus, the finding that the opening of the direct channel raises the retail price of product \(r\) applies only if the restrictive condition on \((c_d, c_r)\) is met. The result does not apply if \(c_d = 0\). In other words, we miss 2. of Proposition 5 if we assume that \(c_d = 0\). Note that the dashed line indicating \(p_{r}^{**} = p_{r}^*\) moves its location to across or below Point A if \(\theta \geq \frac{21}{37} - \frac{6}{37}\sqrt{3}\) \((\approx 0.287)\). Thus, the retail price of product \(r\) unambiguously decreases in \(\tilde{R}_{1b}\) following the opening of the direct channel if \(\theta\) is so large that \(\theta \geq \frac{21}{37} - \frac{6}{37}\sqrt{3}\).

4 Concluding Remarks

In this paper, we developed a vertical differentiation model of an upstream manufacturer and two downstream retailers with cost asymmetry. In the model, the manufacturer produces a physical product and sells it to the downstream retailers. Moreover, the manufacturer can
choose to open a new direct channel for a digital product. We compared the equilibrium outcomes in the models incorporating and omitting a direct digital channel.

The results obtained in this paper are summarized as follows. Opening a direct digital channel increases or decreases the quantity of the physical product depending on the costs of the manufacturer and retailers. When the marginal cost of producing the physical product is sufficiently small, the quantity of the physical product increases following the opening of the direct digital channel. However, the direct digital channel may reduce the quantity of the physical product sold by the inefficient retailer even if it increases total quantity. Opening the direct digital channel may increase the quantity of the physical product sold by the efficient retailer even if it reduces total quantity. These results arise because of cost asymmetry between the two retailers, which is new to the literature to the author’s best knowledge. Moreover, under certain conditions related to the manufacturer’s costs and if the quality of the digital product is low, then an increase in the wholesale price of the physical product by the manufacturer may raise its retail price and may cause the inefficient retailer to cease trading, thereby eliminating retail competition. This lowers the welfare of consumers who continue to purchase the physical product after the new digital product comes onto the market; however, this result applies only under restrictive conditions on the model’s parameters.

Finally, we conclude this paper with some qualifications. First, we assumed that the
quality of the digital product is low when compared with the alternate physical product. However, through its efforts a manufacturer can improve the quality of the digital product. Consequently, the quality of the digital product can be high relative to the physical counterpart, at least for some consumers. In addition, while our models assumed that each consumer buys at most one product, some consumers may actually purchase both the physical and digital product (e.g., a paper book and an e-book). With this in mind, we should include an additional dimension in our vertical differentiation models so we can examine the effect of both quality improvements and product bundling. Second, we ignored competition between manufacturers. However, in many industries, manufacturers develop new products and compete with each other, especially where both the upstream and downstream markets are oligopolies. For example, there is acute competition between manufacturers of computers and similar digital devices. To more fully understand the impact of a new marketing channel for physical or digital products, we must then explicitly consider upstream competition.\footnote{In the recent literature, Calzada and Valletti (2012) is related to the first issue, and Mizuno (2012) tackles the second issue.}

These are important matters that should be considered in future research.
Appendix

Proof of Lemma 1

Suppose each retailer’s quantity is positive in equilibrium. Combining the optimal quantities of Retailer $i$ ($= H, L$) in the second line of (6) yields

$$q_H = \frac{1 + c_L - 2c_H - w}{3},$$

(28)

$$q_L = \frac{1 + c_H - 2c_L - w}{3}.$$  

(29)

For these quantities to be positive, the following conditions must hold:

$$w < 1 + c_L - 2c_H,$$  

(30)

$$w < 1 + c_H - 2c_L.$$  

(31)

Note that (30) is sufficient to satisfy (31). When (30) does not hold, Retailer $H$’s optimal quantity is zero. Then, Retailer $L$ becomes a monopolist in the retail market. The monopolist’s optimal quantity is

$$q_L = \frac{1 - w - c_L}{2}.$$  

(32)

For this quantity to be positive, the following condition must be satisfied:

$$w < 1 - c_L.$$  

(33)

Proof of Lemma 2

From (8), we find that the shape of (9) depends on $w$.

First, suppose that $w$ is in the range $1 + c_L - 2c_H < w < 1 - c_L$. Differentiating $\Pi^m$ with respect to $w$ yields

$$\frac{\partial \Pi^m}{\partial w} = 0 : w = \frac{1}{2} c_r - \frac{1}{2} c_L + \frac{1}{2}.$$  

(34)

It must be the case that

$$1 + c_L - 2c_H < \frac{1}{2} c_r - \frac{1}{2} c_L + \frac{1}{2} < 1 - c_L,$$

or, equivalently,

$$1 - c_L - 4 (c_H - c_L) < c_r < 1 - c_L,$$
for (34) to locally maximize $\Pi^m$ around (34). The second-order condition for a local maximum is satisfied. Substituting (34) into (9) yields the locally maximized profit,

$$\Pi^m_a = \frac{1}{8} (1 - c_L - c_r)^2.$$  \hfill (35)

Next, suppose that $w$ is in the range $c_r < w < 1 + c_L - 2 c_H$. Setting a price below $c_r$ is not profitable. Thus, we only consider this range. Differentiating $\Pi^m$ with respect to $w$ yields

$$\frac{\partial \Pi^m}{\partial w} = 0 : w = \frac{1}{2} c_r - \frac{1}{4} c_L - \frac{1}{4} c_H + \frac{1}{2}. \hfill (36)$$

It must be the case that

$$c_r < \frac{1}{2} c_r - \frac{1}{4} c_L - \frac{1}{4} c_H + \frac{1}{2} < 1 + c_L - 2 c_H,$$

or, equivalently,

$$c_r < \min \left( 1 - \frac{1}{2} c_L - \frac{1}{2} c_H, 1 + \frac{5}{2} c_L - \frac{7}{2} c_H \right) = 1 - c_L - \frac{7}{2} (c_H - c_L),$$

for (36) to locally maximize $\Pi^m$ around (36). The second-order condition for a local maximum is satisfied. Substituting (36) into (9) yields the locally maximized profit,

$$\Pi^m_b = \frac{1}{24} (2 - c_H - c_L - 2 c_r)^2.$$  \hfill (37)

From (35) and (37), we find that

$$\Pi^m_b - \Pi^m_a = \frac{1}{24} \left[ (2 - c_H - c_L - 2 c_r)^2 - 3 (1 - c_L - c_r)^2 \right]$$

$$= \frac{1}{24} \left[ (2 - c_H - c_L - 2 c_r) - \sqrt{3} (1 - c_L - c_r) \right]$$

$$\times \left[ (2 - c_H - c_L - 2 c_r) + \sqrt{3} (1 - c_L - c_r) \right].$$

Therefore,

$$\Pi^m_b > \Pi^m_a \iff c_r < 1 - c_L - \left( \sqrt{3} + 2 \right) (c_H - c_L),$$

and the wholesale price, $w$, that maximizes (9) is as follows:

$$w = \begin{cases} 
\frac{1}{2} c_r - \frac{1}{2} c_L + \frac{1}{2} & \text{if } 1 - c_L - \left( \sqrt{3} + 2 \right) (c_H - c_L) \leq c_r < 1 - c_L, \\
\frac{1}{2} c_r - \frac{1}{4} c_L - \frac{1}{4} c_H + \frac{1}{2} & \text{if } c_r \leq 1 - c_L - \left( \sqrt{3} + 2 \right) (c_H - c_L). 
\end{cases}$$

27
**Proof of Proposition 1**

From Lemma 2, under Assumption 1, the wholesale price maximizing the manufacturer’s profit is

\[ w^* = \frac{1}{2} c_r - \frac{1}{4} c_L - \frac{1}{4} c_H + \frac{1}{2}. \]  

(38)

By substituting (38) into (4), (7) and (8), we obtain the equilibrium retail price, each retailer’s equilibrium quantity and the equilibrium total quantity in the retail market. Substituting (38), the retail price and the corresponding quantities into (5) yields the equilibrium profit of each retailer. The equilibrium profit of the manufacturer is (37).

**Proof of Lemma 3**

We consider separately three cases, which differ in their dependence on the level of total quantity, \( q_H + q_L \). First, we consider the following case:

\[ q_H + q_L < 1 - \frac{p_d}{\theta}. \]  

(39)

In this case, the direct channel generates positive demand. From the second line of (15), the best-response functions for the two retailers are as follows:

\[
q_H = \frac{1 - \theta + p_d - w - c_H - (1 - \theta) q_L}{2 (1 - \theta)},
\]

\[
q_L = \frac{1 - \theta + p_d - w - c_L - (1 - \theta) q_H}{2 (1 - \theta)}.
\]

By assuming that both these quantities are positive in equilibrium, we obtain

\[
q_H = \frac{1 - \theta + c_L - 2 c_H + p_d - w}{3 (1 - \theta)},
\]

(40)

\[
q_L = \frac{1 - \theta + c_H - 2 c_L + p_d - w}{3 (1 - \theta)}.
\]

(41)

For these quantities to be positive, the following conditions must hold:

\[
w < 1 - \theta + c_L - 2 c_H + p_d, \]  

(42)

\[
w < 1 - \theta + c_H - 2 c_L + p_d. \]  

(43)

Note that (42) is sufficient to satisfy (43). By substituting (40) and (41) into (39), we obtain

\[
w > \frac{3 - \theta}{2 \theta} p_d - \frac{1 - \theta + c_H + c_L}{2}. \]  

(44)
When (42) does not hold, Retailer $H$’s optimal quantity is zero. Then, in the retail market, Retailer $L$ becomes a monopolist, whose optimal quantity is

$$q_L = \frac{1 - \theta - c_L + p_d - w}{2(1 - \theta)}.$$  \hfill (45)

For this quantity to be positive, the following condition must be satisfied:

$$w < 1 - \theta - c_L + p_d.$$ \hfill (46)

By substituting $q_H = 0$ and (45) into (39), we obtain

$$w > \frac{2 - \theta}{\theta} p_d - (1 - \theta + c_L).$$ \hfill (47)

To sum up, (40) and (41) are the equilibrium quantities under (42) and (44), and $q_H = 0$ and (45) represent the equilibrium under (46) and (47).

Next, consider the following case:

$$q_H + q_L > 1 - \frac{p_d}{\theta}.$$ \hfill (48)

In this case, the direct channel generates no demand. From the fourth line of (15), the best-response functions for the two retailers are

$$q_H = \frac{1 - w - c_H - q_L}{2},$$

$$q_L = \frac{1 - w - c_L - q_H}{2}.$$  

By assuming that both these quantities are positive in equilibrium, we obtain

$$q_H = \frac{1 + c_L - 2c_H - w}{3},$$ \hfill (49)

$$q_L = \frac{1 + c_L - 2c_L - w}{3}. \hfill (50)$$

For these quantities to be positive, the following conditions must hold:

$$w < 1 + c_L - 2c_H,$$ \hfill (51)

$$w < 1 + c_H - 2c_L.$$ \hfill (52)
Note that (51) is sufficient to satisfy (52). Substituting (49) and (50) into (48) yields

\[ w < \frac{3}{2\theta}p_d - \frac{1 + c_H + c_L}{2}. \]  

(53)

When (51) does not hold, Retailer \( H \)'s optimal quantity is zero. Then, in the retail market, Retailer \( L \) becomes a monopolist, whose optimal quantity is

\[ q_L = \frac{1 - w - c_L}{2}. \]  

(54)

For this quantity to be positive, the following condition must be satisfied:

\[ w < 1 - c_L. \]  

(55)

By substituting \( q_H = 0 \) and (54) into (48), we obtain

\[ w < \frac{2}{\theta}p_d - (1 + c_L). \]  

(56)

To sum up, (49) and (50) are the equilibrium quantities under (51) and (53), and \( q_H = 0 \) and (54) represent the equilibrium under (55) and (56).

Next, consider the following intermediate case:

\[ q_H + q_L = 1 - \frac{p_d}{\theta}. \]  

(57)

In this case, \( \hat{v} = v_{d2} \). No consumer derives a positive surplus from product \( d \). The third lines of (15) and (57) are identical in this case. Any quantities satisfying (57) represent an equilibrium.

**Proof of Lemma 4**

First, suppose that \((p_d, w) \in R_{3a} \cup R_{3b}\). In these regions, \( p_d \) is high enough to eliminate the direct channel demand. Thus, the (local) maximization wholesale price is the same as that described in Lemma 2. Given Assumption 2, the wholesale price that locally maximizes the manufacturer’s profit is not in \( R_{3b} \) but in \( R_{3a} \); this wholesale price is

\[ w = \frac{1}{2}c_r - \frac{1}{4}c_L - \frac{1}{4}c_H + \frac{1}{2}. \]  

(58)

By substituting this price into the inequality \( w \leq \frac{3}{2\theta}p_d - \frac{1 + c_H + c_L}{2} \) (which is one of the
conditions characterizing $R_{3a}$, we obtain

$$p_d \geq \frac{1}{3} \theta \left( \frac{4 + 2c_r + c_H + c_L}{2} \right).$$

That is, setting $p_d$ high enough to satisfy the above inequality is necessary for (58) to be in $R_{3a}$.

Second, suppose that $(p_d, w) \in R_2$. Again, in this region, the direct channel generates no demand, and the manufacturer’s profit is

$$\Pi^m = (w - c_r) \left( 1 - \frac{p_d}{\theta} \right).$$

Thus, the manufacturer can simultaneously increase $w$ and decrease $p_d$ as long as $(p_d, w) \in R_2$. As a result, it is sufficient to consider the pair $(p_d, w)$ along the segment

$$w = \frac{2 - \theta}{\theta} p_d - (1 - \theta + c_L),$$  

(59)

where $p_d$ and $w$ satisfy

$$\frac{\theta (1 - \theta - c_H + c_L)}{1 - \theta} \leq p_d \leq \theta, \quad \frac{1 - \theta - (2 - \theta) c_H + c_L}{1 - \theta} \leq w \leq 1 - c_L$$  

(60)

or along the segment

$$w = \frac{3 - \theta}{2\theta} p_d - \frac{1 - \theta + c_H + c_L}{2},$$  

(61)

where $p_d$ and $w$ satisfy

$$\frac{\theta (1 - \theta + c_H + c_L)}{3 - \theta} \leq p_d \leq \frac{\theta (1 - \theta - c_H + c_L)}{1 - \theta}, \quad 0 \leq w \leq \frac{1 - \theta - (2 - \theta) c_H + c_L}{1 - \theta}.$$  

(62)

(Note that in Figure 1 the direct price corresponds to segment AC, and the wholesale price corresponds to segment CE.)

Consider the former case (59). The manufacturer’s profit is

$$\Pi^m = (w - c_r) Q_r,$$

$$= \left[ \left( \frac{2 - \theta}{\theta} p_d - (1 - \theta + c_L) \right) - c_r \right] \left( 1 - \frac{p_d}{\theta} \right).$$  

(63)
Differentiating (63) with respect to $p_d$ yields
\[ \frac{\partial \Pi^m}{\partial p_d} = 0 : p_d = \frac{\theta (3 - 2\theta + c_L + c_r)}{2 (2 - \theta)}. \] (64)

Substituting (64) into (59) yields
\[ w = \frac{1 - c_L + c_r}{2}. \]

To locally maximize (63), the above pair $(p_d, w)$ must satisfy (60) or, equivalently, must satisfy
\[ 1 - c_L - \frac{2 (2 - \theta)}{1 - \theta} (c_H - c_L) \leq c_r \leq 1 - c_L. \]
The second-order condition for a local maximum is satisfied. Substituting (64) into (63) yields the locally maximized profit,
\[ \Pi^m_F = \frac{1}{2 - \theta} \left( \frac{1 - c_r - c_L}{2} \right)^2. \] (65)

Consider the latter case (61). The manufacturer’s profit is
\[ \Pi^m = (w - c_r) Q_r 
\quad = \left[ \left( \frac{3 - \theta}{2\theta} p_d - \frac{1 - \theta + c_H + c_L}{2} \right) - c_r \right] \left( 1 - \frac{p_d}{\theta} \right). \] (66)

Differentiating (66) with respect to $p_d$ yields
\[ \frac{\partial \Pi^m}{\partial p_d} = 0 : p_d = \frac{\theta (4 - 2\theta + c_H + c_L + 2c_r)}{2 (3 - \theta)}. \] (67)

Substituting (67) into (61) yields
\[ w = \frac{1}{2} c_r - \frac{1}{4} c_L - \frac{1}{4} c_H + \frac{1}{2}. \] (68)

To locally maximize (66), the above pair $(p_d, w)$ must satisfy (62) or, equivalently, must satisfy
\[ \frac{c_H + c_L - 2}{2} \leq c_r \leq 1 - c_L - \frac{(7 - 3\theta) (c_H - c_L)}{2 (1 - \theta)}. \]
The second-order condition for a local maximum is satisfied. Substituting (67) into (66)
yields the locally maximized profit,

$$\Pi_L^m = \frac{[2 - 2c_r - (c_H + c_L)]^2}{8(3 - \theta)}. \tag{69}$$

From (65) and (69), we find that

$$\Pi_F^m - \Pi_L^m = \frac{(2 - 2c_r - (c_H + c_L))^2}{8(3 - \theta)} - \frac{1}{2 - \theta} \left(\frac{1 - c_r - c_L}{2}\right)^2,$$

$$= \frac{1}{2 - \theta} \left[ \sqrt{\frac{2 - \theta}{8(3 - \theta)}} (2 - 2c_r - (c_H + c_L)) - \left(\frac{1 - c_r - c_L}{2}\right) \right],$$

$$\times \left[ \sqrt{\frac{2 - \theta}{8(3 - \theta)}} (2 - 2c_r - (c_H + c_L)) + \left(\frac{1 - c_r - c_L}{2}\right) \right].$$

Therefore,

$$\Pi_L^m - \Pi_F^m > 0 \iff \sqrt{\frac{2 - \theta}{8(3 - \theta)}} (2 - 2c_r - (c_H + c_L)) - \left(\frac{1 - c_r - c_L}{2}\right) > 0,$$

$$\iff c_r < 1 - c_L - \sqrt{\frac{2(2 - \theta)}{2(1 - \theta)}} \left(\sqrt{3 - \theta} + \sqrt{2(2 - \theta)}\right) (c_H - c_L). \tag{70}$$

Note that, for any $\theta \in (0, 1),$

$$\frac{\sqrt{2(2 - \theta)}}{2(1 - \theta)} \left[ \sqrt{3 - \theta} + \sqrt{2(2 - \theta)} \right] < \frac{3(\sqrt{3} + 2) - (\sqrt{3} + 3)\theta}{3(1 - \theta)}.$$

Thus, given Assumption 2, it follows that (70) holds and $\Pi_L^m > \Pi_F^m.$

Third, suppose that $(p_d, w) \in R_{1a} \cup R_{1b}$. In these regions, the direct channel generates positive demand. Consider the case of $R_{1b}$. Differentiating (21) with respect to $w$ and $p_d$ yields

$$\frac{\partial \Pi^m}{\partial w} = 0 : 2w + \theta + c_L + c_d - c_r - 2p_d - 1 = 0, \tag{71}$$

$$\frac{\partial \Pi^m}{\partial p_d} = 0 : (2\theta - 4) p_d + (\theta + 2c_d + 2w\theta - \theta^2 + \theta c_L - \theta c_d - \theta c_r) = 0, \tag{72}$$

respectively. The second-order conditions for local maximization are satisfied as follows:

$$\frac{\partial^2 \Pi^m}{\partial w^2} = -\frac{1}{1 - \theta} < 0, \quad \frac{\partial^2 \Pi^m}{\partial p_d^2} = -\frac{1}{\theta(1 - \theta)} (2 - \theta) < 0,$$
\[
\frac{\partial^2 \Pi^m}{\partial w^2} \frac{\partial^2 \Pi^m}{\partial p_d^2} - \left( \frac{\partial^2 \Pi^m}{\partial w \partial p_d} \right)^2 = \frac{2}{\theta (1 - \theta)} > 0.
\]

Solving (71) and (72) for \((p_d, w)\) yields

\[
w = \frac{1 + c_r - c_L}{2}, \quad \text{(73)}
\]
\[
p_d = \frac{\theta + c_d}{2}. \quad \text{(74)}
\]

For (73) and (74) to be within \(R_{1b}\), it must be the case that

\[
c_r < 1 - \theta - c_L + c_d,
\]
\[
c_r > 1 - \theta + 3c_L - 4c_H + c_d,
\]
\[
c_r > \theta - c_L - 1 + \left( \frac{2}{\theta} - 1 \right) c_d.
\]

By substituting (73) and (74) into (21), we obtain the locally maximized profit,

\[
\Pi_{1b}^m = \frac{(1 - \theta - c_r + c_d - c_L)^2}{8 (1 - \theta)} + \frac{1}{4\theta} (\theta - c_d)^2. \quad \text{(75)}
\]

Next, consider the case of \(R_{1a}\). Differentiating (21) with respect to \(w\) and \(p_d\) yields

\[
\frac{\partial \Pi^m}{\partial w} = 0 : 4w + 2\theta + c_H + c_L + 2c_d - 2c_r - 4p_d - 2 = 0, \quad \text{(76)}
\]
\[
\frac{\partial \Pi^m}{\partial p_d} = 0 : 2 (\theta - 3) p_d + (\theta + 3c_d + 4w\theta - \theta^2 + \theta c_H + \theta c_L - \theta c_d - 2\theta c_r) = 0, \quad \text{(77)}
\]

respectively. The second-order conditions for local maximization are satisfied as follows:

\[
\frac{\partial^2 \Pi^m}{\partial w^2} = -\frac{4}{3(1 - \theta)} < 0, \quad \frac{\partial^2 \Pi^m}{\partial p_d^2} = -\frac{2(3 - \theta)}{3\theta (1 - \theta)} < 0,
\]
\[
\frac{\partial^2 \Pi^m}{\partial w^2} \frac{\partial^2 \Pi^m}{\partial p_d^2} - \left( \frac{\partial^2 \Pi^m}{\partial w \partial p_d} \right)^2 = \frac{8}{3\theta (1 - \theta)} > 0.
\]

Solving (76) and (77) for \((p_d, w)\) yields

\[
w = \frac{2 (1 + c_r) - (c_H + c_L)}{4}, \quad \text{(78)}
\]
\[
p_d = \frac{\theta + c_d}{2}. \quad \text{(79)}
\]
For (78) and (79) to be within \( R_{1a} \), it must be the case that
\[
c_r < 1 - \theta - \frac{c_L + c_H}{2} - 3 (c_H - c_L) + c_d,
\]
\[
c_r > \frac{3 - \theta - c_d}{2\theta} - \frac{1}{2} + \frac{1}{2} - \frac{c_L + c_H}{2}.
\]

By substituting (78) and (79) into (21), we obtain the locally maximized profit,
\[
\Pi^m_{1a} = \frac{[2 - 2\theta - (c_H + c_L) + 2c_d - 2c_r]^2}{24 (1 - \theta)} + \frac{1}{4\theta} (\theta - c_d)^2 .
\]  

(80)

From (75) and (80), we find that
\[
\Pi^m_{1a} - \Pi^m_{1b} = \frac{(2 - 2\theta - (c_H + c_L) + 2c_d - 2c_r)^2}{24 (1 - \theta)} - \frac{(1 - \theta - c_r + c_d - c_L)^2}{8 (1 - \theta)},
\]
\[
= \frac{1}{8 (1 - \theta)} \left[ \frac{2}{\sqrt{3}} \left( 1 - \theta + c_d - c_r - \frac{c_H + c_L}{2} \right) - (1 - \theta + c_d - c_r - c_L) \right] \times \left[ \frac{2}{\sqrt{3}} \left( 1 - \theta + c_d - c_r - \frac{c_H + c_L}{2} \right) + (1 - \theta + c_d - c_r - c_L) \right].
\]

Therefore,
\[
\Pi^m_{1a} \geq \Pi^m_{1b} \iff c_r \leq 1 - \theta - c_L - \left( \sqrt{3} + 2 \right) (c_H - c_L) + c_d.
\]

Now, we are ready to compare the three cases. In the first two cases, the wholesale price, \( w \), that locally maximizes \( \Pi^m \) is the same, and the direct channel generates no demand. Therefore, it is optimal for the manufacturer to lower \( p_d \) as much as possible in order to increase sales from the retail channel. That is, (67) and (68), in combination with (61), maximize the manufacturer’s profit in region \( R_2, R_{3a} \) or \( R_{3b} \).

Next, by comparing (69) and (80), we find that
\[
\Pi^m_{1a} \geq \Pi^m_{L} \iff \frac{\theta + c_d}{2} \leq \frac{\theta (4 - 2\theta + c_H + c_L + 2c_r)}{2 (3 - \theta)},
\]
\[
\iff c_r \geq \left( \frac{3 - \theta}{2\theta} \right) c_d - \frac{1 - \theta + c_H + c_L}{2}.
\]  

(81)

If (81) holds with equality, then the pair of prices maximizing \( \Pi^m \) in \( R_{1a} \) is the same as that maximizing \( \Pi^m \) in \( R_2, R_{3a} \) or \( R_{3b} \). That is, equality in equation (81) represents the border of the region of optimal prices for the manufacturer. When \( c_d \) is sufficiently low that (81) holds with inequality, it is optimal for the manufacturer to lower \( p_d \) to stimulate demand through the direct channel.
Solving the following equations for \((c_d, c_r)\):

\[
c_r = 1 - \theta - c_L - \left(\sqrt{3} + 2\right) (c_H - c_L) + c_d,
\]
\[
c_r = \left(\frac{3 - \theta}{2\theta}\right) c_d - \frac{1 - \theta + c_H + c_L}{2},
\]
yields

\[
c_d = \theta \left[ 1 - \frac{2\sqrt{3} + 3}{3(1 - \theta)} (c_H - c_L) \right], \quad (82)
\]
\[
c_r = 1 - c_L - \frac{3(\sqrt{3} + 2) - (\sqrt{3} + 3) \theta}{3(1 - \theta)} (c_H - c_L). \quad (83)
\]

That is, when \(c_d\) and \(c_r\) satisfy (82) and (83), respectively, the maximized profit of the manufacturer in region \(R_{1b}\) is the same as that in regions \(R_{1a}\) and \(R_2\). Note that (83) corresponds to the positive upper limit on \(c_r\) in Assumption 2. Under this assumption, the following inequality holds:

\[
1 - \frac{2\sqrt{3} + 3}{3(1 - \theta)} (c_H - c_L) > 1 - \frac{2\sqrt{3} + 3}{3(\sqrt{3} + 2) - (\sqrt{3} + 3) \theta} (1 - c_L). \quad (84)
\]

Note that \(0 < \frac{2\sqrt{3}+3}{3(\sqrt{3}+2)-(\sqrt{3}+3)\theta} < 1\) for any \(\theta\) satisfying \(0 < \theta < 1\). Note also that \(0 < 1 - c_L < 1\), which is implied by Assumption 2. Therefore, the right-hand side of (84) is positive. As a result, (82) is positive and less than \(\theta\).

To sum up, the optimal prices for the manufacturer depend on \((c_d, c_r)\) and are summarized as follows:

\[
(p_d, w) = \begin{cases} 
\left(\frac{\theta + c_d}{2}, \frac{2(1 + c_r) - (c_H + c_L)}{4}\right) & \text{if } (c_d, c_r) \in \hat{R}_{1a}, \\
\left(\frac{\theta + c_d}{2}, \frac{1 - c_d + c_r}{2}\right) & \text{if } (c_d, c_r) \in \hat{R}_{1b}, \\
\left(\frac{\theta(4 - 2\theta + c_H + c_L + 2c_r)}{2(3 - \theta)}, \frac{2(1 + c_r) - (c_H + c_L)}{4}\right) & \text{if } (c_d, c_r) \in \hat{R}_2,
\end{cases}
\]
where the regions conditioning \((c_d, c_r)\) are defined as follows:

\[
\hat{R}_{1a} = \left\{(c_d, c_r) \mid \begin{array}{l}
    c_d < \frac{\theta}{3-\theta} + c_H + c_L + 2c_r \\
    c_d \geq c_r + c_L + (\sqrt{3} + 2) (c_H - c_L) - (1 - \theta)
\end{array} \right\},
\]

\[
\hat{R}_{1b} = \left\{(c_d, c_r) \mid \begin{array}{l}
    c_d \leq c_r + c_L + (\sqrt{3} + 2) (c_H - c_L) - (1 - \theta), \\
    c_r < 1 - c_L - \frac{3(\sqrt{3} + 2) - (\sqrt{3} + 3)}{3(1-\theta)} (c_H - c_L)
\end{array} \right\},
\]

\[
\hat{R}_2 = \left\{(c_d, c_r) \mid \begin{array}{l}
    c_d \geq c_r + c_L + (\sqrt{3} + 2) (c_H - c_L) - (1 - \theta) \\
    c_r < 1 - c_L + c_d
\end{array} \right\}.
\]

Proof of Proposition 2

From Lemma 4, if the pair \((c_d, c_r)\) belongs to \(\hat{R}_{1a}, \hat{R}_{1b}\) or \(\hat{R}_2\), then the pair \((p_{d*}, w_{*})\) maximizing the manufacturer’s profit correspondingly falls within \(R_{1a}, R_{1b}\) or \(R_2\). For any one of the three regions \((\hat{R}_{1a}, \hat{R}_{1b}\) or \(\hat{R}_2\)), by substituting the corresponding pair \((p_{d*}, w_{*})\) into (16) and (18)–(20), we obtain equilibrium values of each retailer’s quantity, the total quantity of product \(r\), the retail price of product \(r\) and the total quantity of product \(d\). Then, by substituting \((p_{d*}, w_{*})\) and the corresponding quantities sold by the retailers into (14) for \(i = H, L\), we obtain the equilibrium profit of each retailer. The equilibrium profit levels of the manufacturer in the three regions \((\hat{R}_{1a}, \hat{R}_{1b}\) and \(\hat{R}_2)\) are given by (80), (75) and (69), respectively.

Proof of Proposition 3

Suppose that \(c_r\) is small enough to satisfy Assumption 2. We use Propositions 1 and 2 to analyze three cases, which are distinguished by different levels of \(c_d\). First, consider the case in which the pair \((c_d, c_r)\) belongs to \(\hat{R}_{1a}\). We obtain

\[
Q_{r*}^{**} - Q_r^* = \frac{2\theta}{6(1-\theta)} \left[ \frac{1}{\theta} c_d - c_L - \frac{1}{2} (c_H - c_L) - c_r \right].
\]

Note that in some regions of \(\hat{R}_{1a}\), the following inequalities hold:

\[
\left(\frac{3 - \theta}{2\theta}\right) c_d - \frac{1 - \theta + c_H + c_L}{2} < \frac{1}{\theta} c_d - c_L - \frac{1}{2} (c_H - c_L) < c_d + 1 - \theta - c_L - \left(\sqrt{3} + 2\right) (c_H - c_L).
\]
That is, the pair \((c_d, c_r)\) yielding \(Q_r^{**} = Q_r^\ast\) satisfies the relationship

\[
c_r = \frac{1}{\theta} c_d - c_L - \frac{1}{2} (c_H - c_L),
\]

which is located in \(\hat{R}_{1a}\). Therefore, given Assumption 2, it follows that \(Q_r^{**} > Q_r^\ast\) if and only if \(c_r < \frac{1}{\theta} c_d - c_L - \frac{1}{2} (c_H - c_L)\).

Next, consider the case in which the pair \((c_d, c_r)\) belongs to \(\hat{R}_{1b}\). We obtain

\[
Q_r^{**} - Q_r^\ast = \frac{(1 - 4\theta) c_r + \theta - 1 + (1 - 4\theta) c_L + 2 (c_H - c_L) + 3c_d - 2\theta (c_H - c_L)}{12 (1 - \theta)}.
\]

Suppose that \(\theta = \frac{1}{4}\). Then, it follows that \(Q_r^{**} = Q_r^\ast\) if and only if

\[
c_d = \frac{1}{4} - \frac{1}{2} (c_H - c_L),
\]

which exceeds the value implied by (82) when evaluated at \(\theta = \frac{1}{4}\). That is, we obtain \(Q_r^{**} < Q_r^\ast\) in \(\hat{R}_{1b}\). Next, suppose that \(\theta \neq \frac{1}{4}\). Then, it follows that \(Q_r^{**} = Q_r^\ast\) if and only if

\[
c_r = -\frac{3}{1 - 4\theta} c_d - c_L + \frac{1 - \theta}{1 - 4\theta} [1 - 2 (c_H - c_L)].
\]

If \(\theta < \frac{1}{4}\), then (82) and (83) satisfy

\[
c_r < -\frac{3}{1 - 4\theta} c_d - c_L + \frac{1 - \theta}{1 - 4\theta} [1 - 2 (c_H - c_L)].
\]

That is, for any \((c_d, c_r)\) \(\in \hat{R}_{1b}\), \(Q_r^{**} < Q_r^\ast\). If \(\theta > \frac{1}{4}\), then \(Q_r^{**} < Q_r^\ast\) if and only if

\[
c_r > \frac{3}{4\theta - 1} c_d - c_L + \frac{1 - \theta}{1 - 4\theta} [1 - 2 (c_H - c_L)].
\]

If \(\frac{1}{4} < \theta \leq 3 - \frac{3}{2}\sqrt{3} (\approx 0.402)\), then (82) and (83) satisfy this inequality, and \(Q_r^{**} < Q_r^\ast\) for any \((c_d, c_r)\) \(\in \hat{R}_{1b}\). If \(3 - \frac{3}{2}\sqrt{3} < \theta < 1\), then there is some region of \(\hat{R}_{1a}\) in which the following inequality holds:

\[
c_d + 1 - \theta - c_L - \left(\sqrt{3} + 2\right) (c_H - c_L) < \frac{3}{4\theta - 1} c_d - c_L + \frac{1 - \theta}{1 - 4\theta} [1 - 2 (c_H - c_L)].
\]

That is, the pair \((c_d, c_r)\) yielding \(Q_r^{**} = Q_r^\ast\) satisfies the relationship

\[
c_r = \frac{3}{4\theta - 1} c_d - c_L + \frac{1 - \theta}{1 - 4\theta} [1 - 2 (c_H - c_L)];
\]

38
which is located in $\hat{R}_{1b}$. Therefore, given Assumption 2, it follows that $Q^*_r > Q^*_r$ if and only if $c_r < \frac{3}{4\theta - 1}c_d - c_L + \frac{1}{2(1-\theta)}[1 - 2 (c_H - c_L)]$.

Next, consider the case in which the pair $(c_d, c_r)$ belongs to $\hat{R}_2$. We obtain

$$Q^*_r - Q^*_r = \frac{\theta}{3(3-\theta)} \left[ 1 - c_L - \frac{1}{2} (c_H - c_L) - c_r \right],$$

which is positive under Assumption 2 because

$$\frac{3(\sqrt{3} + 2) - (\sqrt{3} + 3)}{3(1-\theta)} - \frac{1}{2} = \frac{(3-\theta)}{6(1-\theta)} \left( 2\sqrt{3} + 3 \right) > 0.$$

**Proof of Proposition 4**

Suppose that $c_r$ is small enough to satisfy Assumption 2. In addition, suppose that the pair $(c_d, c_r)$ belongs to $\hat{R}_{1a}$. First, consider the quantity levels of Retailer $H$ in the presence and absence of a direct digital channel. From Propositions 1 and 2, we obtain

$$q^*_H - q^*_H = \frac{2\theta}{12(1-\theta)} \left[ \frac{1}{\theta} c_d - c_L - \frac{7}{2} (c_H - c_L) - c_r \right].$$

Note that there are regions of $\hat{R}_{1a}$ in which the following inequalities hold:

$$\left( \frac{3 - \theta}{2\theta} \right) c_d - \frac{1 - \theta + c_H + c_L}{2} < \frac{1}{\theta} c_d - c_L - \frac{7}{2} (c_H - c_L)$$

$$< c_d + 1 - \theta - c_L - \left( \sqrt{3} + 2 \right) (c_H - c_L).$$

That is, the pair $(c_d, c_r)$ yielding $q^*_H = q^*_H$ satisfies the relationship

$$c_r = \frac{1}{\theta} c_d - c_L - \frac{7}{2} (c_H - c_L),$$

which is located in $\hat{R}_{1a}$. Therefore, given Assumption 2, it follows that $q^*_H > q^*_H$ if and only if $c_r < \frac{1}{\theta} c_d - c_L - \frac{7}{2} (c_H - c_L)$.

Next, consider the quantity sold by Retailer $L$ in the presence and absence of a direct digital channel. From Propositions 1 and 2, we obtain

$$q^*_L - q^*_L = \frac{2\theta}{12(1-\theta)} \left[ \frac{1}{\theta} c_d - c_L + \frac{5}{2} (c_H - c_L) - c_r \right].$$
Note that there are regions of \( \hat{R}_{1a} \) in which the following inequalities hold:

\[
\left( \frac{3 - \theta}{2\theta} \right) \cd - \frac{1 - \theta + cH + cL}{2} < \frac{1}{\theta} \cd - cL + \frac{5}{2} (cH - cL)
\]

\[
< \cd + 1 - \theta - cL - \left( \sqrt{3} + 2 \right) (cH - cL).
\]

That is, the pair \((c_d, c_r)\) yielding \(q^*_L = q^*_r\) satisfies the relationship

\[
c_r = \frac{1}{\theta} \cd - cL + \frac{5}{2} (cH - cL),
\]

which is located in \( \hat{R}_{1a} \). Therefore, given Assumption 2, it follows that \(q^*_L > q^*_r\) if and only if \(c_r < \frac{1}{\theta} \cd - cL + \frac{5}{2} (cH - cL)\).

**Proof of Proposition 5**

Suppose that \(c_r\) is small enough to satisfy Assumption 2. We use Propositions 1 and 2 to analyze three cases, each characterized by a different level of \(c_d\).

First, consider the case in which the pair \((c_d, c_r)\) belongs to \(\hat{R}_{1a}\). We obtain

\[
p^*_r - p^*_r = -\frac{1}{6} (\theta - \cd),
\]

which is negative in \(\hat{R}_{1a}\).

Next, consider the case in which the pair \((c_d, c_r)\) belongs to \(\hat{R}_{1b}\). We obtain

\[
p^*_r - p^*_r = \frac{1}{12} [1 - 3\theta - cL - 2 (cH - cL) + 3\cd - c_r].
\]

Thus, \(p^*_r = p^*_r\) if and only if

\[
c_r = 1 - 3\theta - cL - 2 (cH - cL) + 3\cd.
\]

Equations (82) and (83) satisfy

\[
c_r \geq 1 - 3\theta - cL - 2 (cH - cL) + 3\cd
\]

if and only if \(\theta \geq \frac{31}{37} - \frac{6}{37} \sqrt{3} (\approx 0.287)\). That is, for any \((c_d, c_r) \in \hat{R}_{1b}, p^*_r < p^*_r\) if \(\theta \geq \frac{31}{37} - \frac{6}{37} \sqrt{3}\).

If \(\theta < \frac{31}{37} - \frac{6}{37} \sqrt{3}\), there is a region of \(\hat{R}_{1b}\) in which the following inequality holds:

\[
c_d + 1 - \theta - cL - \left( \sqrt{3} + 2 \right) (cH - cL) < 1 - 3\theta - cL - 2 (cH - cL) + 3\cd.
\]
That is, the pair \((c_d, c_r)\) yielding \(p_r^{**} = p_r^*\) satisfies the relationship

\[ c_r = 1 - 3\theta - c_L - 2(c_H - c_L) + 3c_d, \]

which is located in \(\hat{R}_{1b}\). Therefore, given Assumption 2, it follows that \(p_r^{**} > p_r^*\) if and only if

\[ c_r < 1 - 3\theta - c_L - 2(c_H - c_L) + 3c_d. \]

Next, consider the case in which the pair \((c_d, c_r)\) belongs to \(\hat{R}_2\). We obtain

\[ p_r^{**} - p_r^* = -\frac{\theta}{3(3 - \theta)} \left[ 1 - c_L - \frac{1}{2}(c_H - c_L) - c_r \right], \]

which is negative under Assumption 2 because

\[ \frac{3(\sqrt{3} + 2) - (\sqrt{3} + 3)\theta}{3(1 - \theta)} - \frac{1}{2} = \frac{(3 - \theta)}{6(1 - \theta)} \left(2\sqrt{3} + 3\right) > 0. \]
References


